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ABSTRACT

This report summarizes the work of a two-year project which focused primarily on the problems that students have with algebra in general, and graphs in particular. The first of two major sections in the document deals with the use of computer software to assist in the teaching of graphing. It concludes that thoughtful design and use of graphing software presents new opportunities for teaching about graphing. The next section of the report centers on the development of research instruments that are intended to study scale in the context of graphs of function. It includes a set of problem-based teaching materials that were used as research tools. The appendices contain descriptions of probes designed to see if students can interpret and create graphs of real-world phenomena, along with instruments dealing with mapping, scale, and computer explorations. (TW)

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MATHEMATICAL, TECHNICAL, AND PEDAGOGICAL CHALLENGES IN THE GRAPHICAL REPRESENTATION OF FUNCTIONS

Technical Report
February 1988

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MATHEMATICAL, TECHNICAL, AND PEDAGOGICAL CHALLENGES
IN THE GRAPHICAL REPRESENTATION OF FUNCTIONS*

Technical Report
February 1988

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I. INTRODUCTION

HISTORICAL PERSPECTIVE

Software that links symbolic and graphical representations of function is proliferating for at least three reasons:

- There is a perceived need to increase the emphasis on graphing in the curriculum (e.g., Fey, 1984).
- It is theoretically reasonable that appropriate visual representations help invest meaning in, and thereby promote the learning of, the symbol system with which algebra students must cope (Kaput, 1986; Kaput, in press).
- Computer technology lends itself well to this application.

Common sense supports the notion that multiple representation can aid understanding. Used thoughtfully, multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation. Said another way, each well-chosen representation conveys part of the meaning best: together, they should improve the fidelity of the whole message. Finally, translation across representations can help to reduce the isolation of each mathematical lesson and help to provide a more coherent and unified view of mathematical method and content. Many student errors may be directly traceable to the use of one or another representation system in isolation, where there is no ready "check" on the validity of actions taken (Kaput, in press).

These are reasonable theoretical arguments, but until the recent proliferation of software based on these ideas, it has been impractical to examine them clinically. Little is actually known about the cognitive impact of multiple linked representation in algebra. While potentially reducing ambiguity, multiple representation also presents a student with more places to look and is potentially complicating and distracting. Introduction of visual thinking into a subject that had relied almost solely on syntactic skills may change who the successful students are, admitting new ones—a desirable effect—but potentially excluding some old ones—an effect of dubious merit.

When we first proposed to study the effects of multiple linked representation on algebra learning, we conceived of the research broadly as a two-part intervention study. In each study, the intervention would consist of a teaching experiment in which students would work with selected software and curricula.

1 One study would analyze the effects of that experience on the eradication of certain well-known classes of algebra manipulation errors. This was to be done first, in part because it seemed to be a straightforward, focussed, applied statistical study with a predictable (and relatively short) time-table and, in part, because it would likely yield data that would help to focus the second study.

2 A second study would explore the impact of this qualitatively different experience with algebra on student mathematical thinking. When the proposal was written, this was not specified in detail because it was neither clear what dimensions might emerge as the

most important to consider nor even whether to focus our attention on effects that would be visible immediately during the algebra course. After all, the experience might plant seeds that remain essentially dormant until some future time, for example when the student enters calculus.

Both of these are intervention questions concerning how students change under a particular treatment. Our first two months of work brought greater definition to these early questions, but also pointed out the great need for fundamental research into the ways that students perceive and interpret graphs.

Our very earliest experiments showed that students often significantly misinterpreted what they saw in graphic representations of function. Left alone to experiment, students could induce rules that were wrong. This finding alone seemed of considerable practical import to software developers and curriculum designers. But it also suggested to us the need for close monitoring of *individuals* engaged in the tasks of interest. An intervention study focussing on classes might well miss essential features of student interaction with this kind of software, features that would tell us *how* the software or curriculum succeeded (if it did) or *why* it failed (if it did).

For these reasons, we changed the course of our study, postponing our intervention-oriented questions while we pursued this prerequisite line of investigation—*how students develop the perceptual and semantic processes they use in interpreting graphical information.* The major products of the first year are a "lay-of-the-land" view of the graphics-related issues that come up when studying algebra in a computer context, and a strategy for studying students' understanding of scale. The first of these is found in section II of this report (beginning on page 5). Our strategy for studying scale underlies next year's research. It can be found in section IV and the appendixes (p. 34 ff.).

Broadly, we hoped to gain insight into how students come to use graphical information to give meaning to symbolic representations of function and vice versa. What, in fact, do students attend to when they look at graphs? What are the misconceptions that they bring with them and how do these misconceptions distort the information that they glean from the graphs? What is the impact of these distortions on student ability to use graphs to inform their understanding and manipulation of symbolic representations of function? How resistant are these misconceptions to instructional correction? All of these questions derive straightforwardly from the larger question: What is the impact of multiple representation on the learning of algebra?

As our work progressed, however, key issues began to multiply and it began to seem as if they were not so easily separable from the multiple-representation question. In fact, while graphing was always around, it was not practical to use very much of it along with symbolic representation until computers and graphing software became widely available in schools. It is therefore not multiple-representation, but multiple-representation *in a computer environment* that becomes the subject of study, and that raises a host of human factors issues connected specifically with the computer. These will be discussed in detail later, but it is important at the outset to know the range of topics that cannot be avoided once computer graphing is part of the curriculum: students' understanding and use of scale and scale information; issues involving concepts of *discrete* and *continuous* quantities;

ways in which semantic distinctions between modeling and mapping affect students' interpretation of functions; clarity about the role of the variable.

Out of this seemingly focused notion of the "development of the perceptual and semantic processes students use in interpreting graphical information" a world of research questions emerged that was vastly too broad for us to pursue and so we narrowed our research topic even further.

One issue—understanding of scale—is practically forced upon the student by the realities of computer graphing. Graphing is done in a space of certain dimensions: to graph within that space, one must choose a scale and an origin. Most classroom graphing that is done by hand on paper begins with functions that fit the paper "window" reasonably with little attention to scale issues. If scale must be considered, one controls it oneself and most likely knows what one wants soon after the first or second computation of the function's value. On the computer, things are different. If students are encouraged to explore freely to see what the graphs of various functions look like, the likelihood of their chosen functions fitting well in any fixed window is considerably lowered. There are two ways out: either the computer must choose a scale on which to display the graph, or the student must do so (perhaps with the aid of the computer). In both cases, if the student is to derive meaning from the visual presentation, it becomes essential that the student understand the effect of the chosen scale on that image.

For the last half of the first research year, all of our questions revolved around scale issues, although, as will be brought out more clearly below, even then we were forced to attend to other questions.

REVIEW OF THE STATE OF THE LITERATURE

Even having narrowed our focus to issues of scale, we could not escape paying attention to other issues insofar as they influenced our ability to interpret what we observed about students' understanding of scale.

At the outset, we thought it necessary to distinguish two kinds of coordinate graphs: 1) graphs in which the independent variable is either time itself or a function of time that is strongly perceived as *causing* the change in the dependent variable; and 2) graphs in which the relation between the variables is an arbitrary mapping. The distinction is useful because it is so reasonable to assume that the presence or absence of real-world referential meaning for a graph will influence the strategies that the student invokes in interpreting the graph. Interpretation of scale may be influenced by expectations about how graphic representations of functions that *model* some phenomenon (e.g., a graph of height against time) relate to aspects of the physical *reality* of the phenomenon itself (e.g., a photograph of a trajectory). For example, students may care more about how a graph looks when it represents a familiar physical phenomenon than when it represents an arbitrary mapping. Furthermore, some transformations (e.g., inverse) become meaningless when the independent variable is time.

As it turned out, this distinction did not help us much in reviewing the literature: although there is a *teaching* literature, very little *research* is reported in either category. On time/causality graphs the research is sparse and on graphs of algebraic functions—the area of most direct relevance to our current work—it is almost non-existent. In addition to those researchers involved directly in this project (in particular, J. Kaput, J. Schwartz, and M. Yerushalmy), we have pursued the work of others: work done by several researchers at The Weizmann Institute of Science (e.g., Dreyfus and Eisenberg, 1987; Zehavi, et al., 1987); at the Shell Centre for Mathematical Education at the University of Nottingham (e.g., Janvier, 1978; and the 1987 curriculum unit *The Language of Functions and Graphs*); at or through Technical Education Research Centers (TERC), Cambridge, Massachusetts (e.g., Barclay, 1987; Brasell, 1986; Mokros and Tinker, 1986); and work done by J. Clement and colleagues (e.g., Clement, 1985; Schultz, Clement and Mokros, 1986 draft); work by Dugdale and her colleagues at the University of Illinois (e.g., Dugdale, 1982); by Alan Osborne (personal communications); by Andee Rubin and her colleagues at Bolt Beranek and Newman, Inc., Cambridge, Massachusetts (personal communications); by G. J. Hitch and colleagues (e.g., Hitch, et al., 1983); and by researchers at The University of California at Berkeley (e.g., work by Laurie Edwards and others cited in Schoenfeld, 1987).

Of the research that we have found so far, only that done at The Weizmann Institute deals directly with the influence of graphs on students' understandings of arbitrary functions. Essentially all of the rest deals with time/causality graphs.

None of the research literature addresses issues of scale (though some of the development work done by Rubin and her colleagues in the area of statistics does attack some problems involved in *teaching* about scale).

METHODS

The observations and research strategies produced in the first year, and described in the next two sections, were developed through the work of a group which included six high school teachers from the greater Boston area, two graduate students, a mathematician specializing in algebra, the project manager, and the project leader, a specialist in mathematics education. The group's purpose was to discuss problems students have with algebra in general and graphs in particular, to plan research activities, to review and interpret results of these activities, and to help guide the still evolving goals of the project. The group served an additional and important function in providing informal and anecdotal reports of student work, teacher biases and expectations, and teaching techniques that had been tried, all of which shed considerable light on our planned direct observations of students' work.

Among the earliest research activities was an analysis of the work of two second-year algebra students using an early prototype of the *Function Analyzer*, a component of *The Algebra Toolkit*, a software series developed at Education Development Center, Inc. (EDC). Our detailed analysis of the full transcript of the interview with these students led us to shift the focus of our research from the impact of multiple representation software on algebra errors to a study of how students perceive and interpret graphs, with particular interest in three issues:

- the meaning of discrete points in the graph
- students' understanding and misunderstanding of the significance of scale
- the interplay between visual and analytic strategies for interpreting graphs

Follow-up studies were designed to narrow this field further. These included several small studies of beginning algebra students (6th through 9th grades) graphing functions and data on distorted graph papers, a larger study (roughly 130 students from 6th through 12th grades) replicating and reexamining work done by Schultz, Clement, and Mokros (1986), and many observations both current and retrospective of student graphing errors. All pointed to the centrality of scale issues in student misunderstanding, misperception, or misuse of graphs.

The recollections and expectations of the thoughtful and experienced teachers in the research group suggested observations that were then performed either by the teachers in the course of their teaching or by the research staff in specially arranged sessions with students. We then discussed these observations as a group. As our discussions shifted course, influenced by the most current questions raised by our observations, we allowed our subsequent explorations to shift course accordingly. This left us with a difficult task in synthesizing a cohesive whole out of our observations and discussions, but was the necessary approach at a time when it was essential that our study remain broad and exploratory.

II. MATHEMATICS, METAPHORS, AND HUMAN FACTORS

Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions*

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Newton, MA

Are graphs of functions more accessible to students than symbolic presentations? Apparently not. Although students have spent years perceiving and drawing visual forms before they first encounter algebraic symbols, graphs have conventions and ambiguities of their own. Some strategies that suffice for interpreting real-world scenes are inappropriate when dealing with the infinite and relatively featureless objects in graphs of polynomials. To interpret graphs correctly, we need mathematical knowledge and expectations, not just perceptual experience. Graphing software intended for educating students who lack such mathematical knowledge must therefore include features not normally built into graphers used primarily for non-educational purposes.

What *do* students see when they look at graphs of functions? Aided by a particularly flexible tool, the *Function Analyzer*, our clinical studies catalogue some differences between student perceptions and those of mathematically knowledgeable adults.

Two conclusions stand out. Simplistic software design or thoughtless use of computer graphing in classrooms may further obscure some of what we already find very difficult to teach. On the other hand, thoughtful design and use of graphing software presents new opportunities to focus on challenging and important mathematical issues that were always important to our students but that were never so accessible before.

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* The research reported here was conducted by the Center for Learning Technology, Education Development Center, Inc. under a subcontract from the Educational Technology Center, Harvard Graduate School of Education, and was funded in part by the United States Office of Educational Research and Improvement (Contract # OERI 400-83-0041). The author wishes to thank colleagues at EDC and ETC for their substantial intellectual contributions to this paper: Wayne Harvey, James Kaput, Marc Konvisser, Philip G. Lewis, Judah Schwartz, Richard J. Umiker, Jane West, and Philip Zodhiates. I also wish to thank EDC for allowing the research group to use an early prototype of the *Function Analyzer*, one component of EDC's *Algebra Toolkit*, in conducting this research. The *Algebra Toolkit* was conceived by Judah Schwartz and Michal Yerushalmy at EDC, and was developed by Wayne Harvey, June Mark, Mike Pappas, and others at EDC. It will be published by Sunburst Communications.

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INTRODUCTION

Software that links symbolic and graphical representations of function is proliferating for at least three reasons:

- There is a perceived need to increase the emphasis on graphing in the curriculum (e.g., Fey, 1984).
- It is theoretically reasonable that appropriate visual representations help invest meaning in, and thereby promote the learning of, the symbol system with which algebra students must cope (Kaput, 1986; Kaput, in press).
- Computer technology lends itself well to this application.

Common sense supports the notion that multiple representation will aid understanding. Used thoughtfully, multiple linked representations increase redundancy and thus can reduce ambiguities that might be present in any single representation. Said another way, each well-chosen representation conveys part of the meaning best: together, they should improve the fidelity of the whole message. Translation across multiple representations can help to reduce the isolation of each mathematical lesson and help to provide a more coherent and unified view of mathematical method and content. Many student errors may be directly traceable to the use of one or another representation system in isolation, where there is no ready "check" on the validity of actions taken (Kaput, in press).

However, the fact is that very little is known about what actually takes place when multiple linked representation software is used extensively in the algebra classroom. Until quite recently with the proliferation of graphing software in schools, it has been impractical to examine the effects systematically.

Our own observations and those of others convince us that there certainly is much benefit to be gained from the *thoughtful use* of this kind of software, and that it must be taken very seriously by teachers and curriculum developers. Proper use of visual imagery gave students new depth and clarity in thinking about old problems. The mathematical richness in linking graphical and symbolic representations of functions also gave students opportunities to pose and explore new problems.

On the other hand, our research has also made it clear that graphing software that is developed along conventional lines—software that may seem perfectly adequate at first—can blur or obscure concepts of great importance. Furthermore, the uninformed classroom use of *any* graphing software brings with it a number of pitfalls. *"Thoughtful use"* becomes the key.

While potentially reducing ambiguity, multiple representation also presents a student with more places to look and is potentially complicating and distracting. Introduction of visual thinking into a subject that had relied almost solely on syntactic skills may change who the successful students are, admitting new ones—a desirable effect—but potentially excluding some old ones—an effect of dubious merit. Furthermore, features not connected *idea* of multiple representation but inextricably bound up in its *computer ion* may also influence how the student learns.

Our earliest experiments showed that students often made significant misinterpretations of what they saw in graphic representations of function. Left alone to experiment, they could induce rules that were misleading or downright wrong. This finding alone is of considerable practical importance to software developers and curriculum designers. Our later observations have generated a long list of questions which must be answered in order to design better software and to make good use of it in a thoughtful curriculum.

This report illustrates and explains the issues that have arisen from our work, focussing mostly on students' observed *mis*interpretations of graphs and on the potential pitfalls of the uninformed development or use of multiple representation software.

MAKING GOOD USE OF VISUAL IMAGERY

Sometimes, the availability of one representation can aid a student in manipulations performed with another representation. For example, good use of graphic representations can aid a student in performing an algebraic computation.

A common elementary algebra error is illustrated by the following reconstructed protocol of a student's work:

Problem: Find the value of x in the equation $4x - 17 = 3x - 4 + x$

Student's approach:	Gather like terms	$4x - 17 = 4x - 4$
	Subtract like amounts	$-4x \qquad -4x$
	(intermediate result)	$0 - 17 = 0 - 4$
	(result)	$-17 = -4$

If a student stopped here and declared that no solution was possible, the most we might complain about is the rigidity with which the student performed the last, unnecessary step. But many students continue even at this point to apply the rules that they have learned.

(result)	$-17 = -4$
Add like amounts	$+4 = +4$
(result)	$-13 = 0$

Again, there is a chance for students to stop and take note of the contradiction, but even here, some students continue, undaunted, to conclude that "-13 is the only number left, so that must be what x equals" and they write down

(conclusion) $x = -13$

By contrast, in a classroom experiment that was reported to us, students who had learned to visualize the graphic representations of each symbolic expression had no such problems (Michal Yerushalmy, personal communication). These students had become used to interpreting equations like

$$4x - 17 = 3x - 4 + x$$

as statements that two functions

$$f(x) = 4x - 17 \quad \text{and} \quad g(x) = 4x - 4$$

were equal. Because these students also had a strong visual sense of these functions, they declared the problem insoluble as soon as they saw the form $4x - 17 = 4x - 4$. Their explanation was straightforward: the lines representing *f* and *g* are parallel, and so there is no point on one that is also on the other. There is, in other words, no place where the lines cross and therefore no *x* at which the two function values are equal.

MAKING BAD USE OF VISUAL IMAGERY

It is tempting to assume that the world of shapes and visual gestalts is much more naturally interpretable than the world of algebraic symbols. It is especially tempting to make this assumption of novices to algebra. After all, students have been seeing shapes and interpreting them since they were born, and only began dealing with the formal notational system of algebra quite recently. Most classroom observations suggest another possibility: namely that graphic representations of function are no more naturally interpretable than are the symbolic representations. They have their own rhetorical conventions that students must learn, and they contain ambiguities that are clarified only through the use of further specifications, such as those contained in the conventional algebraic notation.

A concrete example will help to show how this can be. Two bright, successful, second year algebra students were shown the computer screen illustrated in figure 1, and asked to discover the polynomial ($-2x^2 + 30x - 108$) that created this graph. They were encouraged to use whatever means they chose, including making various computer-supported measurements on the graph and trying out various expressions and observing differences between the graphs they created and the target at which they were aiming.

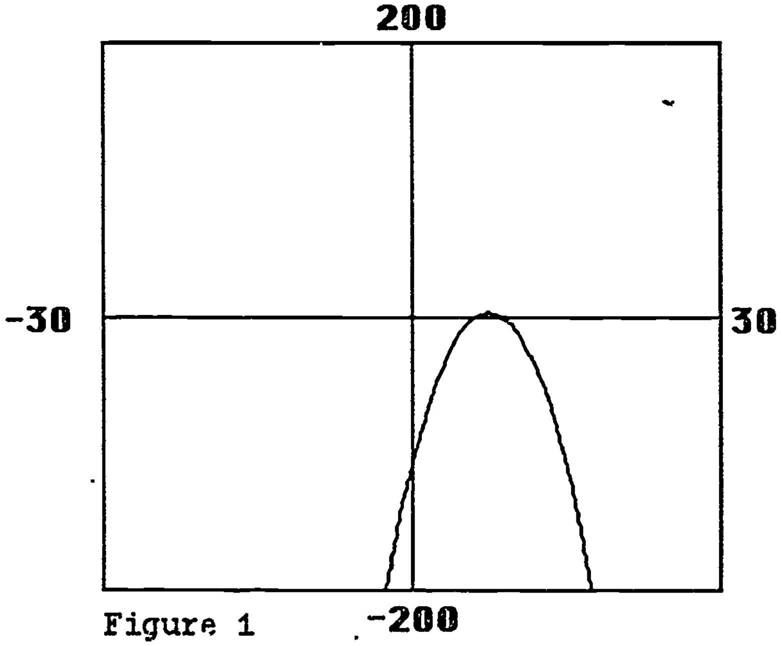


Figure 1

Although the students had never previously tried to match a target graph, they had had some prior experience with graphing software—exploring values and scale with an early prototype of the *Function Analyzer* component of EDC's *Algebra Toolkit* and two instructional sequences developed for this research project—and they had built up some expectations about the effect of the constant term and the coefficient of x^2 in the graph of a parabola. Appropriately, their first analyses made use of these notions.

They had a well-developed notion that something they referred to as "shape" was controlled by the coefficient of x^2. They knew that if the parabola was "upside down," the coefficient must be negative, and after a single experiment they reasoned from the "pointiness" of the parabola that the x^2 coefficient might be around -2. They also had a notion of "height" and believed that it was controlled by the constant term. The measure of "height" that they chose was the y-intercept, which they estimated to be about -100. Indicating that they had no idea what to use for the coefficient of x, they arbitrarily picked 2.

Figure 2 shows the target function and the function that they designed: $-2x^2 + 2x - 100$. Their analysis had been excellent. Their parabola appeared to have exactly the same "shape" and, as well as one could see at this scale, the same y-intercept as the parabola they were trying to match. Only the x coefficient remained undiscovered. Yet, even though they had chosen the y-intercept as the measure of "height"—and that, according to that measure, they had done quite well—the compelling perception that their parabola is lower than the target seduced them into redefining "height." The visual impression so dismayed them that they capitulated and "corrected" the constant term from -100 to +4 to account for their "error." They made this choice despite both their reasoned argument about the value of the constant term and their outright admission that their choice for the coefficient of the linear term was totally arbitrary. The new expression, $-2x^2 + 2x + 4$, was in some ways further from the target than the first, yet it was more satisfying to them because it was just as "high" as the graph they were aiming at (figure 3).

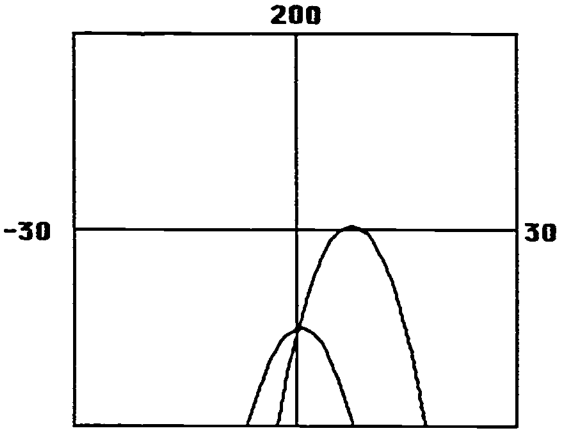


Figure 2

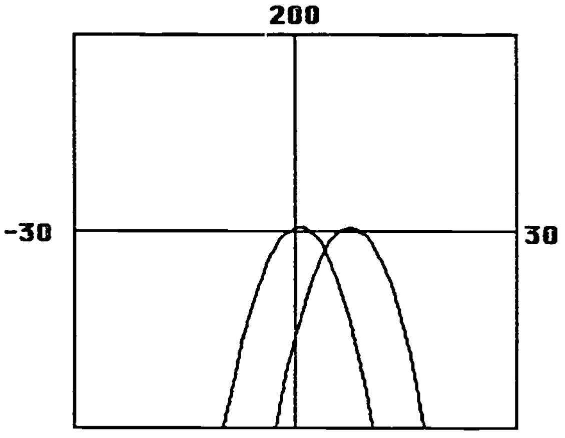


Figure 3

The way in which the illusion distracted them from their originally correct analysis of the problem is reminiscent of the not-quite-conserver in the familiar Piagetian task in which equal quantities of juice are poured into glasses that differ in width. Initially two identical glasses are filled with identical quantities of juice and the child verifies that they are the same. When the juice from one of these glasses is then poured into a narrower container, its level rises higher than the level in the other original glass. Young children's thinking in this situation seems dominated by the visual impression: the new glass must have more. Older children and adults witnessing this experiment are guided more strongly by their expectation that quantity remains invariant despite appearances. In between, there is an intriguing stage when a child might well *expect* that after pouring from one container to the other the amounts would be the same, but would then give in to the perception—even spontaneously expressing surprise, just as the algebra students did. Logical thinking at this stage has developed considerably, but is not robust enough to win over perception.

In our case, the students' confusion appeared to result from a shift of attention from one feature (the *y*-intercept) to another (the overall "height" as suggested by the height of the vertex). In other cases, the confusion seems to arise from the same mechanisms that give rise to well-known perceptual illusions. Consider, for example, what happens when these two students take the expression they have just developed and begin to change the coefficient of x to "move it over." Figure 4 shows how the graphs appear when they have all but the constant term correct. When the students looked at graphs like this, they knew that they needed to adjust the constant term, but they also remarked that the inner parabola looked more obtuse than the outer. (In this example, the target parabola appears blunter than the upper one.) As was true of their confusion regarding the meaning of "height," illusions such as the one illustrated here were sometimes powerful enough to draw their attention back to the already correct coefficient of the x^2 term and cause them to change it.

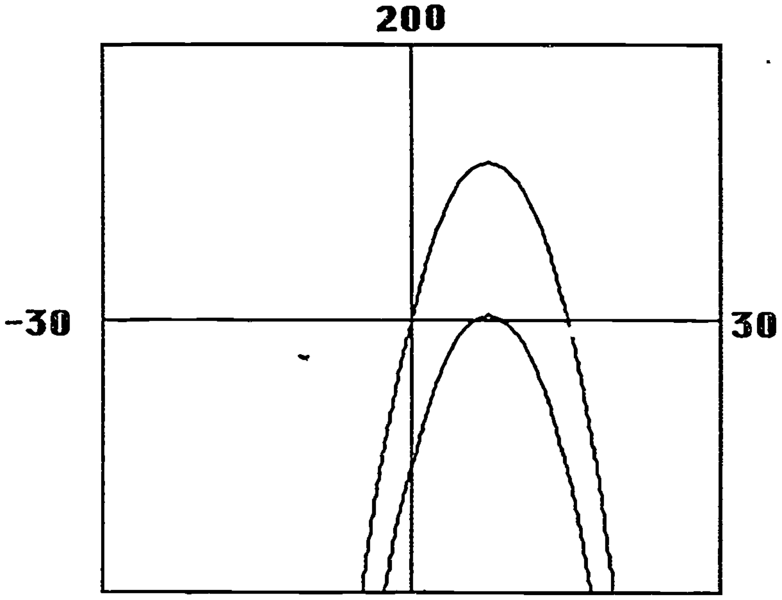


Figure 4 $-2x^2+30x+4$

A general theory of the interpretation of graphs must take context into consideration. In our explanation of the students' difficulties, we have treated their interpretation as if it were conditioned only by the visual presentation and some semantic processing that lumps *y*-intercept in with "height." In fact, that is not the only part of the problem situation that may have affected their approach. In their symbolic representation of the parabola, they were required to use a specific *form* —the unfactored polynomial, ax^2+bx+c. From their comments, it became clear that they ascribed three attributes to the graph of any parabola:

1 Its orientation (up or down)
2 Its pointiness
3 Its position, with the sub-attributes horizontal position and vertical position.

These might well have been assigned in some order to the parameters a, b, and c, but the students were already aware that A controlled a kind of superattribute, "shape," with the sign controlling orientation and the magnitude controlling pointiness. That left *b* and *c* to control position. It seems only natural to assign the two remaining parameters to the two sub-attributes of position, which would encourage the reinterpretation of the function of *c* from *y*-intercept to "height" and would then, by default, leave horizontal control to *b*. Had the students been trying to fill in a different symbolic form—in particular, had their form been something like $a(x-h)^2+v$ (or if the variable *x* in the coefficient form of the polynomial could have been replaced by $\alpha x'-\beta$)—the parameters would have more closely fit their expectations. In such a case, their problem solving behavior may not have appeared so confused and their concepts may have seemed less fragile.

TOWARD A GENERAL THEORY OF INTERPRETATION OF GRAPHS

The Cartesian graph spaces we see in books and on computer screens are rectangular sections of a plane on which some shape appears. Generally, only a portion of the shape appears because the domain on which the function is defined is usually the entire real axis. Through our experiences with partial views of real objects (e.g., views of things being shifted up or down as viewed through a window) we develop working strategies for interpreting such views. As we first learn to read graphs, we interpret what we see in them according to strategies that have been successful for us in other realms, and we continue to use such strategies until our new experiences teach us to do otherwise.

Our everyday strategies fail when we try to interpret objects that are infinite in size and relatively poor in discrete, qualitative features. What we experience is often a perceptual or attentional illusion. The student work described above gives examples of both.

Imagine a person slowly descending on a scaffold outside your office window. As the person's feet first appear at the top of your window, you already have a very good idea of the overall shape of the person. Assuming a constant rate of descent, you have a good idea when that person will be fully visible. Aided in part by the availability of readily identifiable, discrete elements in the scene (e.g., shoelaces, buttons), you have no difficulty at all knowing which direction (down) the person is moving. Finally, because people are not too variable in size (among other clues) you know that it is a 6-foot person descending immediately outside your window and not a 240 foot tall person descending 40 feet away.

In the graph interpretation situation, things are very different. Overall shape, magnitude of a translation (corresponding to the rate of descent of the person in the window), direction of movement, and scale (corresponding to the distance of the person viewed in the window) may all become ambiguous when the object viewed is unbounded in size and has a smooth shape devoid of readily recognizable sub-elements like buttons and shoelaces.

The sources of illusions

Some of the ambiguities of graphs appear to arise from processes akin to well-known visual illusions. Included among the causes of these illusions are the interaction between the position and orientation of the graph and the shape of its window, and the interaction between the scale of the graph and the scale of the window.

Consider, for example, what one sees when looking at a family of linear functions $ax+b$ that differ only in the value of the constant b. If we already know the algebra, we have built up some analytic expectations. What we expect to see is that the graph of a line moves up as b is increased, as in figures 5 and 6.

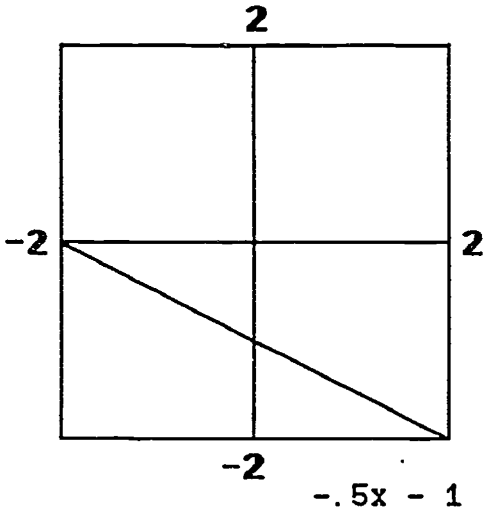


Figure 5

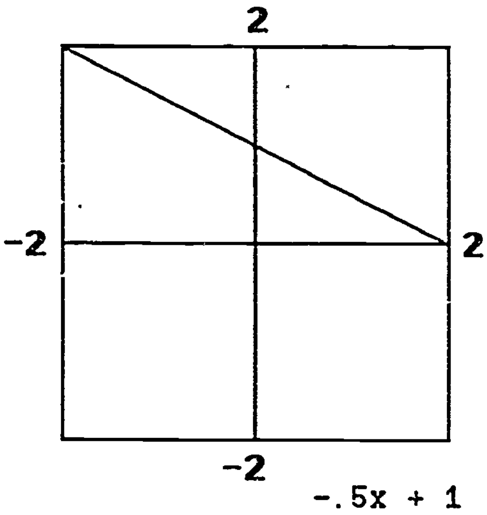


Figure 6

Because an infinite line presents us with no discrete points to watch, however, it may also appear to be moving from left to right as the constant term increases (figures 7 and 8) or even from right to left if the line slope is positive. The way the line appears to move depends on the angle it makes with the window through which you view it and on the shape of that window. Though the appearance is a perceptual phenomenon—not one that any amount of algebraic sophistication can change—algebraic sophistication can lead us to ignore appearances. We may even be able to "see" that the segment of line visible in figure 7 has "moved off the top of figure 8" and a new segment, previously unseen, is now visible.

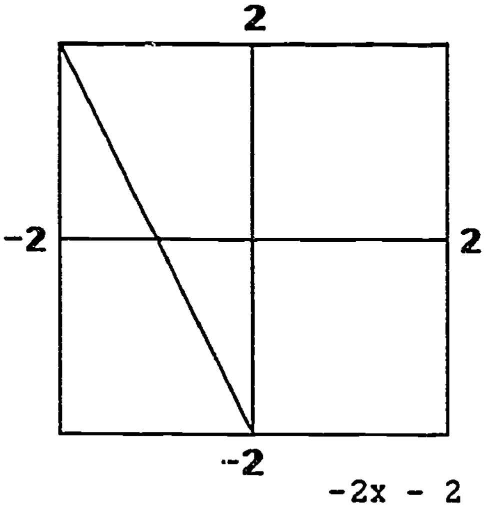


Figure 7

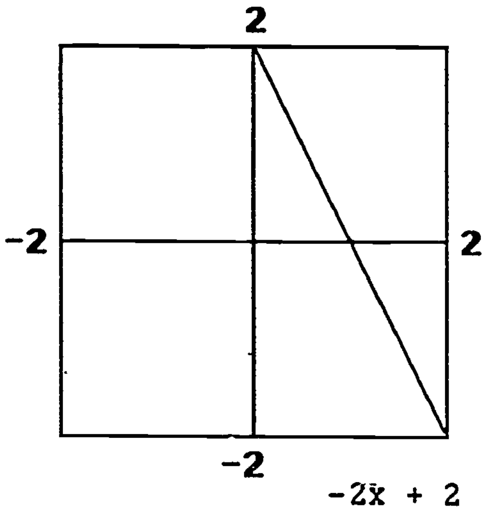


Figure 8

Beginners lack such algebraic sophistication and need some discrete points to watch—tic marks and labels on the graph as the mathematical version of buttons and shoelaces—if they are to interpret the graph in a useful and unambiguous way. The addition of one or more points—e.g., $f(0)$ and $f(1)$—helps the student trace the movement of the entire graph. In this case, $f(0)$ would appear at the bottom of figure 7 and the top of figure 8, providing at least one visual cue that pulls for a bottom-to-top rather than left-to-right interpretation.

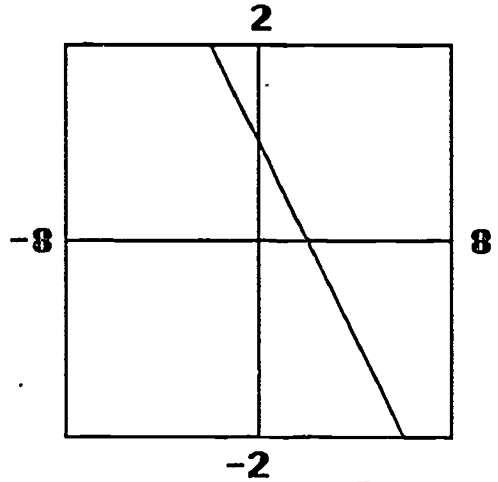
Because a novice lacks the analytic expectations that are needed to see through the illusion, graphing packages that lack "button-and-shoelace" enhancements can be treacherous. Unconstrained inductive learning experiences with such software may certainly lead students to "correct" conclusions, but they may also lead to very complex rules like:

> There are five cases that describe how the graph of a linear function $ax+b$ changes as b increases.

$\lvert a\rvert \ll 1$	the line moves up as b increases
$\lvert a\rvert \gg 1,\ a > 0$	the line moves to the left
$\lvert a\rvert \gg 1,\ a < 0$	the line moves to the right
$\lvert a\rvert \approx 1,\ a < 0$	the line moves diagonally to the northeast
$\lvert a\rvert \approx 1,\ a > 0$	the line moves diagonally to the northwest

Such complex formulations may be the stimulus for valuable mathematical thinking and discussion, but they are certainly not what we want students to learn as "facts." Worse yet, student propensity to choose positive integer values rather than decimals or negatives for coefficients—therefore missing the cases where $|a| < 1$—means that most students develop the left-right theory without even seeing the other movements that might lead some of them to expect or want a simplification. Put simply, a wrong theory is the most likely result of the casual introduction of an inductive learning experience such as this into a curriculum that is not otherwise designed to make use of the questions such an experience raises.

There is an added complication. Even the complex rule given above assumes that x and y are symmetric on the graph. The (visual) angle that a line makes with its "window" depends both on the line's (mathematical) slope and on the relationship of the scales of the two axes. In figures 5-8, the x and y axes are represented in the same scale. Figure 9 represents the same function that appears in figure 6 but, because the scale has been changed, its graph resembles the family of functions represented in figures 7 and 8.



-.5x + 1 Figure 9

Remarkably, even this does not fully capture the complexity of the perceptual situation. The shape of the window is involved in the interpretation of positional relationship among figures inside the window. The parallel lines in figure 10 appear to be one above the other, whereas the parallel lines in figure 11 appear to lie next to one another. Furthermore, the shape of the window has influenced our perception of the slopes of the lines: in reality all four lines form the same angle with the horizontalbut many people perceive the lines in the taller window to be less sloped. A similar phenomenon can be seen in figures 12 and 13.

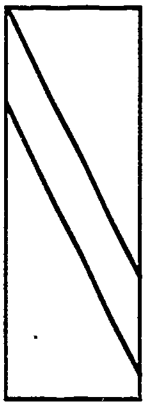


Figure 10

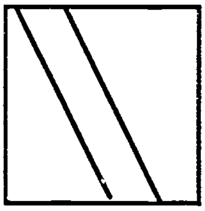


Figure 11

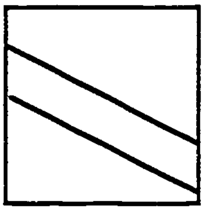


Figure 12

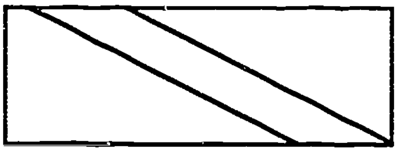


Figure 13

Scale affects perception in other ways as well. An infinite line viewed close up (figure 14) or from afar (figure 15) appears not to change shape, though it moves "closer" to the center of the window. This accords perfectly with our everyday experience with normal objects: as we veiw an (ordinary) object from the same direction but at varying distances, angles in the object are preserved but distances (in this case, the distance from the center of the window) are not. (Of course, the line in figure 15 may equally well be perceived as "higher," suggesting that there has been a change in the constant term.)

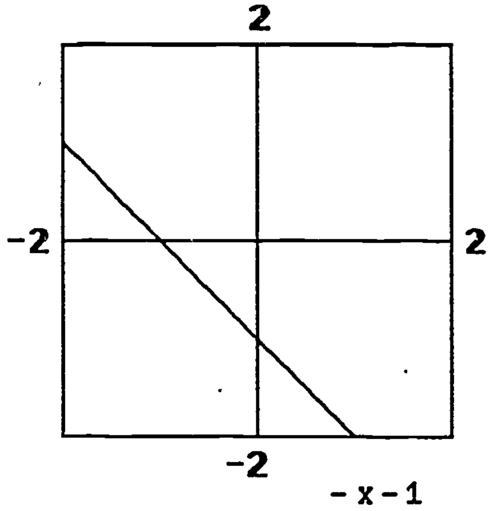


Figure 14

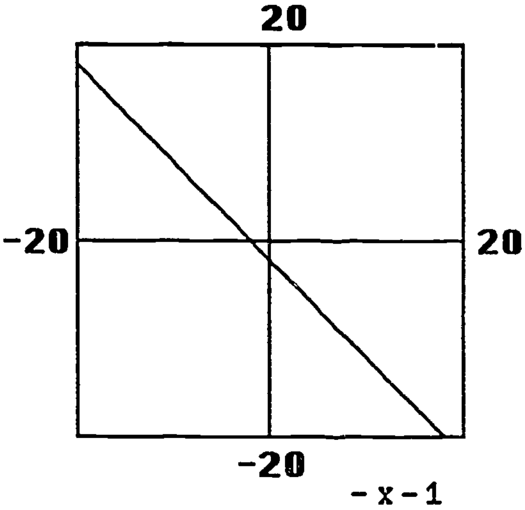


Figure 15

We have a very different experience with the parabola. Viewed close up (figure 16) and from afar (figure 17), the parabola does appear to change shape. Although the portion of the plane that we see in figure 16 is only 1/100th of that shown in figure 17, we automatically compare the tiny chunk in figure 16 with all of figure 17, fooling our eyes into thinking that angle has not been preserved.

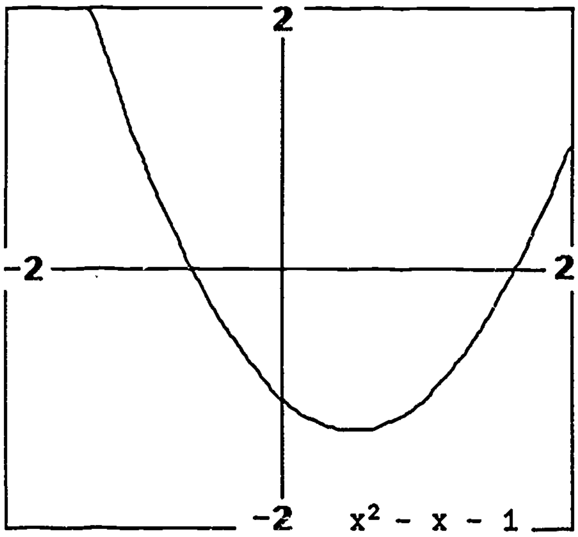


Figure 16

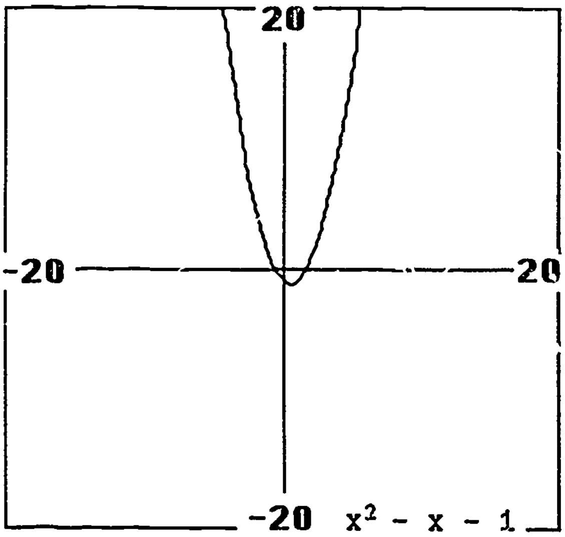


Figure 17

If we compare equivalent portions, we avoid the illusion. Figure 18 shows what portion of figure 17 is illustrated in figure 16.

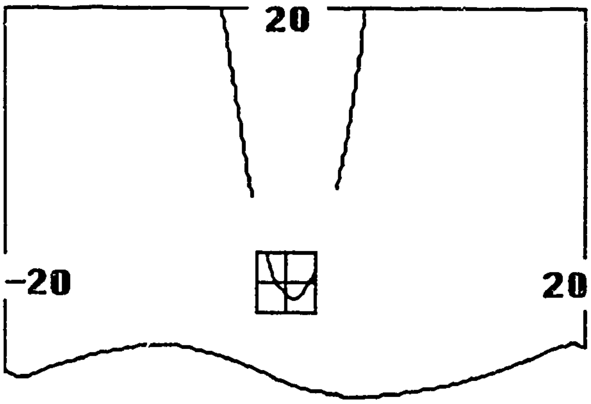


Figure 18

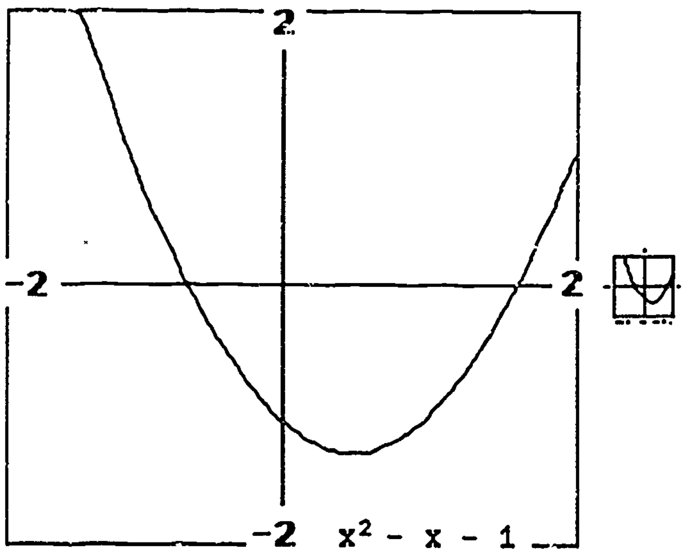


Figure 19

Figure 19 compares figure 16 to a 90% reduction of itself. This time, the two parabolas appear to be the same shape. Thus, when the scale of the parabola changes in a window of fixed size, the parabola appears to change shape. Yet, when the scale of the *window* changes along with the scale of the object in it—that is, when we see the window as well as the parabola from afar—we have no such illusion.

Understanding this interaction between scale and "shape" is important because students typically use "shape" of a parabola (on a constant scale and in a fixed-size window) to determine the *A* coefficient. Thus, though they learn strategies for solving their problem, the strategies are based on an underlying notion—that parabolas may have different shapes—that is erroneous. The shape that they see is, in part, an artifact of scale.

Just as *scale* can alter our perception of the shape of a parabola, so may *position*. In figures 20 and 21, the upper parabola appears blunter. Even though in both figures the upper and lower parabolas can be superimposed through simple translation, students are sometimes tempted to adjust the x^2 coefficient as well.

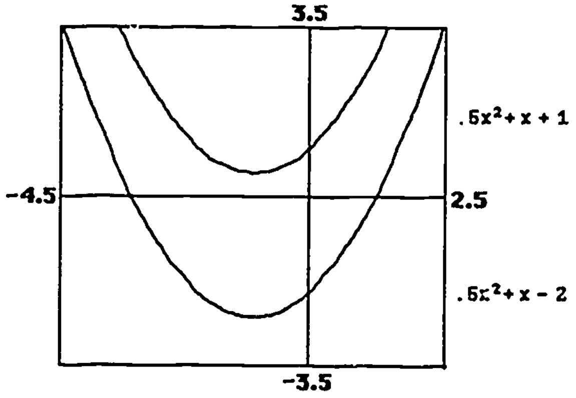


Figure 20

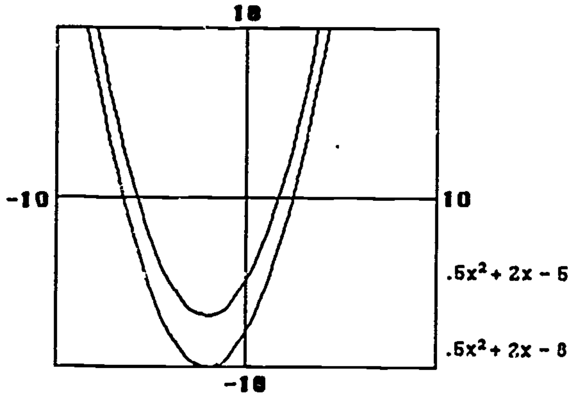


Figure 21

As with the illusions illustrated in figures 5-13, this is a perceptual phenomenon, not a function of one's expertise with algebra. Experts know where to look and what to ignore in order to see beyond the illusion. For non-experts, enhancements of the graphic presentation are needed in order to reduce the ambiguity of the image. The ability to mark points on the curve and the use of the window frame as a scale reference are two such enhancements mentioned above.

Knowing that conventional graphs (e.g., ones that lack specially marked points) *invite* misinterpretation—and knowing even what directions the misinterpretations are likely to take—obliges us to develop less ambiguous presentations. We can eliminate many of the misinterpretations quite easily. We can also milk the mathematical richness of some of these misinterpretations by providing class time for discussion of students' observations and conjectures. However, *studying* these misinterpretations has turned out not to be so straightforward. Though they may often be avoided by manipulating a single graphical feature, they appear to be grounded in a complex interaction of student beliefs and attitudes.

Scale as a focal issue

By the time we had made these observations, it had already become apparent that we should be focusing most of our attention on issues related to scale. We came to this decision in part because so many of the illusions we had observed had to do with misunderstanding, misperception, or misuse of scale information. But there was another reason, as well. The whole question of the use of computer software that links graphical and symbolic representations of function practically forces the issue.

Graphing is done in a space of certain dimensions: to graph within that space, one must choose a scale and an origin. Most classroom graphing that is done by hand on paper begins with functions that fit the paper "window" reasonably with little attention to scale issues. If scale must be considered, one controls it oneself and most likely knows what one wants soon after the first or second computation of the function's value. On the computer, things are different. If students are encouraged to explore freely to see what the graphs of various functions look like, the likelihood of their chosen functions fitting well in any fixed window is considerably lowered. There are two ways out: either the computer must choose a scale on which to display the graph, or the student must do so (perhaps with the aid of the computer). In both cases, if the student is to derive meaning from the visual presentation, it becomes essential that the student understand the effect of the chosen scale on that image.

Despite our early commitment to focus our attention on scale, the near total lack of literature on scale issues and the very sparse literature on any aspects of interpretation of linked graphical and symbolic representations of functions left us feeling a need to test our own techniques and interpretations by getting as close as we could to those few experiments and data on student graphing that had previously been reported. We began by repeating part of an experiment devised by Schultz, Clement, and Mokros (1986), giving some of their test items to ten classes of students from sixth through twelfth grade (about 130 students in all). The test items required students to draw or interpret graphs of familiar phenomena, such as the speed of a bicycle going over a hill.

The total number of errors made by all of our students was very small and did not fall clearly into the types reported in the original study, but those few errors did fit a pattern that, once again, confirmed that scale issues were particularly confusing for students. Many of the non-conventional representations that students used were what we came to call "event charts," in which neither horizontal nor vertical axes were used in a strictly interval way. In general, the horizontal axis was laid out as an ordinal scale in time: sequential, but without necessarily evenly spaced intervals. The vertical axis was often purely nominal, listing events that took place over time but not treating them even as an ordinal scale, let alone an interval scale.

The work shown in figure 22 is representative of these event charts. At the top of the figure is the diagram (borrowed directly from Schultz, Clement, and Mokros, 1986) that had been presented to the students. It shows the progress (from left to right) of a bicyclist riding on a level, then up a hill, across the level top, down the other side, and then on ground level again. Students were asked to use the axes below the diagram, initially blank axes, to graph the speed of the cyclist during the time intervals indicated. One student's work is shown on these axes. She labeled the horizontal axis with the letters *a* through *j*, placing the *a* at the leftmost point and the *j* at the rightmost, and spacing them roughly as they were spaced on the diagram above (preserving some of the interval character of that scale). She labeled the vertical axis with the same descriptive terms that were used on the diagram, but sorted these terms in some way that seemed meaningful to her. In doing this, she created something of an ordinal scale, though the order is idiosyncratic. (Other students performed no sorting at all, repeating labels on the vertical axis whenever they reappeared on the diagram!) Each heavy dot on her chart (except the last two) represents the *onset* of one of the conditions described in the diagram. Thus, the dot at *c* marks the onset of "constant slow speed," and the dot at *d* marks the onset of "speeding up." The line segments that connect each pair of dots serve to highlight the contour and make the drawing look like a line graph, but the segments themselves contain no information.

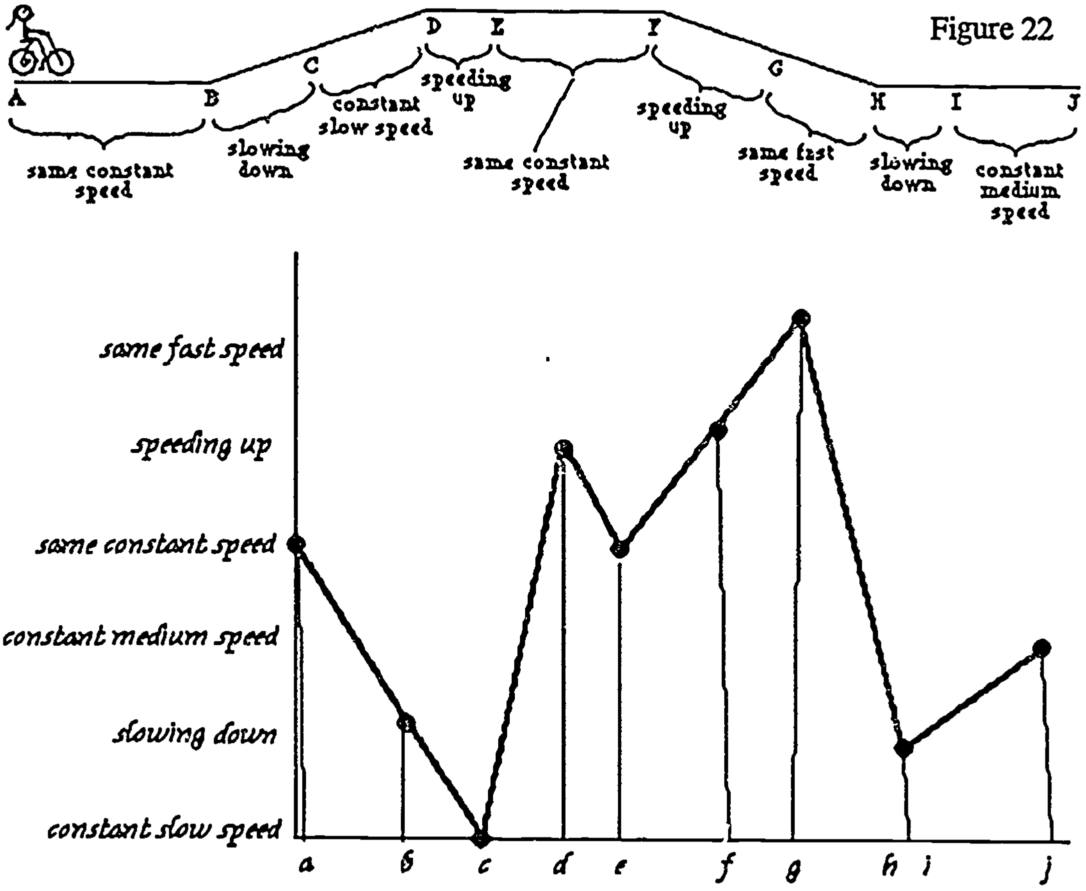


Figure 22

Although this student has created a chart of events, her chart is not as purely qualitative as some others. She shows evidence that she is attending to some quantitative as well as qualitative relationships. As we already noted when we discussed her labeling of the axes, this student creates a kind of ordinality out of the purely nominal labels she assembles along the vertical axis. She even defers slightly to the actual speeds likely to be attained by the cyclist—the speeding up that she represents at *f* is slightly higher than what she shows at *d*. But her drawing shows no sense of continuous change over time. Likewise, her horizontal axis shows an attempt to express quantitative relationships despite her ultimate failure to do so. Her spacing of the points along the horizontal axis resembles an interval scale, as does her insistance on spreading her representation from the first point (*a*) to the last point (*j*), yet when she tries to use this scale, she attaches the information she is representing to the points themselves and not to the intervals between them. The conflict eventually requires her to compromise and leave one point floating between *h* and *i*.

Interactions as a complicating factor

Despite our decision to focus our attention on scale, it became apparent early in the course of our work that a general theory of the interpretation of graphs would have to contend with a variety of issues that interact in complex ways. Even in the course of investigating scale alone, we would not be able to protect ourselves from having to attend to some of those other issues that might confound our observations. These other matters provided a kind of contextual support for our research that, especially in the absence of supporting literature, was absolutely essential.

Perceptual features and cognitive coding

Earlier, we described the phenomenon of students basing a computation correctly on the *y*-intercept of a graph but then using a different feature—the apparent "height" of the resulting graph—to judge the correctness of their computation. This shift of attention from one feature to another led the students to reject their correct computation and use another that was farther from their goal. This confusion seems to arise from students performing one act but coding it mentally as another and being "caught" by the most salient perceptual feature. Students' interpretations of a graph are often inappropriately influenced by such irrelevant perceptual features as the angle at which a linear function intersects the frame of the graph. They also have a tendency to judge the distance between two curves (and therefore various translations from one to the other) in a direction roughly normal to the bisector of the perceived angle between the curves, again influenced by certain features of the frame in which the curve appears. The fact that such perceptual features can be so misleading means that we must be thoughtful about our interpretations of failures at scaling tasks. While such failures may certainly be indicators of conceptual weaknesses on scale issues, perceptual distractions from the task may also be the source of some errors. Like scale, this phenomenon appears to be unresearched.

The concept of variable

With a suitable image (e.g., a rubber-sheet), scale distortions of geometric figures or pictures may be understandable to students, but algebraic interpretations of similar distortions of algebraic curves require not only an understanding of the scale change, but of the meaning of the variable and of the parameters of the function that surround it (e.g., coefficients in a polynomial). To the naive student, there may appear to be little difference between the a, b, c and x in the definition $f(x)=ax^2+bx+c$ (cf. Kuchemann, 1978, and Wagner, 1981). They are all letters, and the level of abstraction—the fact that a is used in place of any *specific* value that it might assume—may even suggest that all of these letters may assume various values. Yet when we speak of the variable in this function, we refer only to x. Software that graphs automatically from left to right, sweeps over a domain in x without any student involvement, and leaves the student active control only over the function's parameters. In other words, the *variable* is not variable by the student but the *constants* are. In effect, then, students who explore the effects on the graph of the function $f(x)=ax^2+bx+c$ as they vary the values of a, b, and c are really studying not $f(x)$, but some *function-valued function*, $f_1(a,b,c)$, whose variables are a, b, and c, and whose value is the graphical output. It is well attested that the concept of variable is difficult for students to learn, at least in static, inert media (Clement, et al., 1981; Wagner, 1981; Clement, 1982). Because the variable and the constants switch roles in many graphing packages, the unthinking use of such software may further obscure rather than clarify this difficult concept. There may also be an appropriate software design response: to make the plotting of x values (e.g., in a table or as pairs on a graph) an explicit action that is different from the changing of parameters.

Identifying discrete points in continuous functions

A curious phenomenon that we have not had the opportunity to study was reported to us by one teacher and was recognized by several others.

A group of students had been taught two methods for solving simultaneous equations. One method was strictly tabular. Students listed numerous pairs for each function and looked for the element that was common to both tables. The second method began similarly, but augmented the search for a common element by plotting each pair on a graph and using the linear pattern of the two sets of points to help locate the single point that was common to both lines. When the students could perform these procedures reliably and it seemed that they understood them well, their teacher taught them the slope-intercept method as a shortcut for plotting a line. No longer did they have to evaluate each function several times in order to plot the points to get a pair of lines from which they could then determine the intersection; they could simply mark a dot representing the constant term on the y-axis and then count one space to the right and move up or down as directed by the coefficient of the linear term. Remarkably, this shortcut appeared to undermine the same students' ability to identify the intersection point *apparently because they had lost the sense that the visually apparent intersection qualified as a real point!* Asked where the two lines intersected, they would no longer report coordinates but rather would point or gesture. It appeared that when they plotted several points and then drew a connecting line, they sensed that there were many *other* points on the line that they might also have plotted. Often, the intersection point had been among the points they plotted themselves, but even if it had not been, they appeared to sense that *it might have been.* On the other hand, slope-intercept lines seemed to be perceived as if they contained *only two points*, the intercept and the one other point that they derived from a run-rise maneuver. This may well hark back to elementary school geometry notions that points exist only where we have *put* them, and that points are *dots* that we have made. Casual language in high-school geometry may suggest the same idea.

A clean understanding of the interaction between scale changes and transformations of a function requires students to have a clear sense of how individual points move under the transformation. (Further discussion of this issue appears below with figures 27 and 28 and their surrounding text.) Again, graphing software that sweeps in a continuous fashion from left to right for all graphs may obscure the rather different image one might have of a function that maps one set of numbers to another.

Computer magic

The potential problems mentioned above all assume that the students *think*, even if only naively, about the graphs. However, there is also the possibility that a novice to graphs will dismiss graphs completely as "how the computer behaves when you type *x*s to it." If the connection between the analytic representation of the function and its graphical representation is perceived as magical or arbitrary, the two representations cannot inform each other. As with each of the other potential problems, this one might be forestalled by appropriate interventions—in this case, all that seems needed is to maintain the kinds of graphing-by-hand experience students currently get and not replace all that experience with computer graphing—but what should *not* be done is to assume that there is no issue here to think about. The simple introduction of computer speed into the production of graphs is a non-trivial intervention that is likely to have significant implications for the curriculum, or for additional changes in the ways such graphing processes can be controlled.

A LIST OF ISSUES AND EXAMPLES

Function and variable

In the service of interpreting our observations of students graphing, we have looked at some of the kinds of difficulties students experience with function and variable.

Skipping over values to ordered pairs of values

We may describe a function as a mapping that relates each element of one set (not necessarily of numbers) to some member of another set (which may or may not be identical with the first set). Thus, the function $f(x) = x^2$ maps values of x (integers, rationals, or however else we may choose to limit the set) to other values (which, in this case, belong to the same set.) Such a description suggests an image for functions that might be

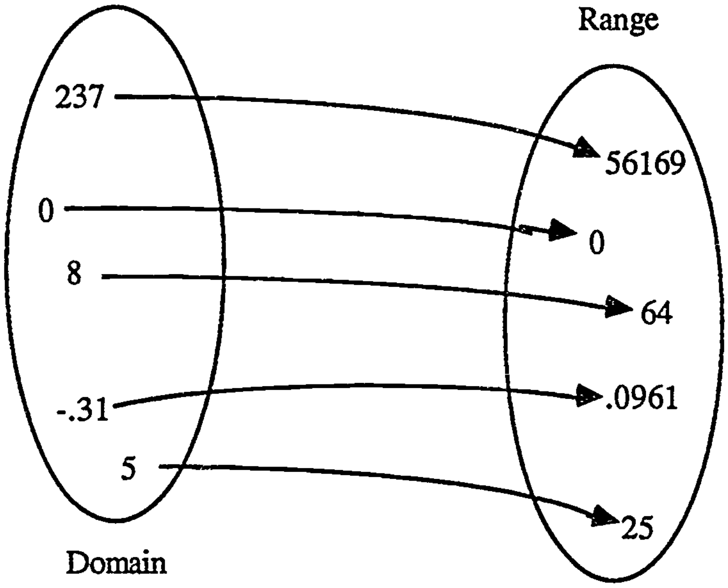


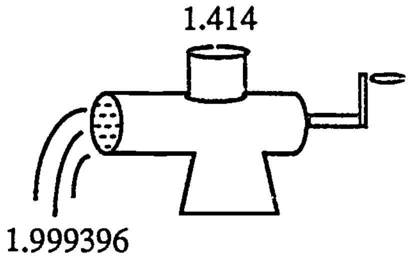
Figure 23

Of course, the image cannot be complete: not all the numbers will fit in the boundaries. But the general idea is clear: each member of one set is matched to an appropriate member of the other set. There is no implied orderliness of the members in the domain set, just a requirement that all of them are there and all of them have arrows from them to some member of the range set.

A tabular representation has much the same imagery. Usually, we also impose additional structure on the table by writing the domain elements in some systematic, orderly way, but that is not an essential feature.

x	→	x^2
1	→	1
2	→	4
3	→	9
4	→	16
5	→	25
6	→	36

Function machine imagery also suggests that a single value come out for each value put in. All of these images suggest *number-valued* functions: put a number in and get a number out.



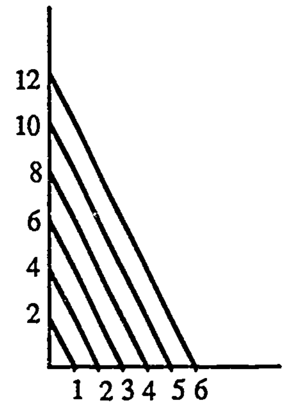


Figure 24

Although traditional graphing may also be seen as a visual representation of the mapping from a single numeric value to another single numeric value, the image it suggests is quite different from the previous ones. It is not uncommon to see students draw a graph like this one (figure 24) when asked to graph $y=2x$. In such a graph, the student is drawing lines to connect the values, in much the way that the arrows in figure 23 were drawn.

By contrast, the traditional graph represents each arrow, along with its associated input and output values, as a point. The compactness of this representation is clearly confusing to some students. We cannot blame the arrow imagery for the kind of graph seen in figure 24, because not all students who draw this sort of representation have seen the set-to-set arrow maps. Instead, the fact that some students seem to create this representation spontaneously suggests that students understand the value-to-value mapping implied by a table and are attempting to represent that image in the graph. The conventional graph, on the other hand, appears to represent quite a different mapping. Where we might describe the table as associating each input *number* to a unique output *number*, graphing seems to associate each input value and its associated output to a unique *point*, as if the function were not a number-in-number-out function, but a number-in-*pair/point*-out function. The process of locating the point by an over-and-up maneuver may tend to support an image of mapping from a *number* on the x-axis to a *point* above it. (This relates to an observation that, for beginning students, points seem to remain identified with the processes by which they are located rather than assuming first class status as mathematical objects in their own right. More will be said about this later.) Thus, students may be viewing their graphs as if they are mappings from numbers to points (figure 25), and may therefore be confused because the tables from which such graphs are constructed do not resemble the table shown in figure 26. If this does explain part of the confusion, we must then clarify for students the variety and nature of inputs and outputs of functions and be explicit about type for each function we use. This certainly asks for greater precision in our explanations and examples, and suggests ideas for additional software support as well.

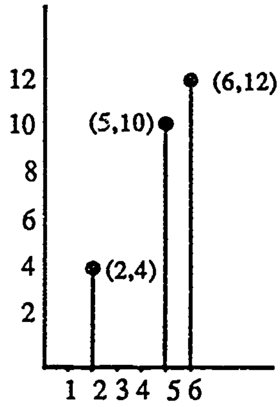


Figure 25

x	→	(point)
1	→	(1,2)
2	→	(2,4)
3	→	(3,6)
4	→	(4,8)
5	→	(5,10)
6	→	(6,12)

Figure 26

Graphs such as the one shown in figure 24 are sometimes explained by invoking general theories of cognitive development: young children have more difficulty coordinating two independent characteristics of an object than in centering their attention on one of them. In this case, the specification of a point in terms of two orthogonal components, and the notation of those two components as an ordered pair of numbers are new notions and are more complex than specifying a single counting number.[1]

An ordered pair of numbers certainly *is* a more complex object than a single one of those numbers by itself, but this may not be the source of students' difficulties. An alternative, non-developmental explanation should be researched. It *may* be that the difficulty lies not so much with *ability* as with *expectation*. Students who have played Battleship™ (or who have walked the streets of Chicago) may fully well recognize that a point may be specified by two coordinates, but may not connect that notion with their *number*-to-*number* representation of function. If this explanation holds, it would relate this error to the already mentioned difficulty students seem to have in recognizing that there are discrete points on a continuous line, despite knowing that they could "put" a point along that line anywhere they might choose. This issue will recur yet again when we look at students' need for an understanding of the infinite.

Skipping over functions to families of functions

A common goal in algebra texts is to draw the students' attention to the influence of the constant term in polynomial functions such as $f(x) = mx + b$, or $g(x) = ax^2 + bx + c$. If the student is working only at an analytic level, only with the algebra and not with the graph, the effect of the constant might most straightforwardly be described as "the value of the function when $x = 0$." When graphic representation is used, the term "*y*-intercept" might be introduced, though students may not necessarily recognize that "*y*-axis" and "$x = 0$" specify the same place.

When automated graphing makes it possible to ask students to induce the effect of the constant by performing many graphing experiments, attention is drawn to the graph as a whole. We speak of a function having a set of values, and the addition of a constant to that function as raising or lowering *each of those values* by the same amount. The student, we hope, shifts attention from a local feature (*y*-intercept) to a global one (vertical translation).

More generally, we can ask students to interpret various transformations on functions, introducing the notion of functions of functions. Dreyfus and Eisenberg (1987), for example, wanted their students to be able to observe a given (and perhaps even totally arbitrary) graph of $f(x)$ and match each of several transformations of that graph with expressions like the following:

---

[1] We have heard these explanations given, sometimes by teachers, but the paucity of research literature on graphing leaves us without any citable researcher's opinion. Arguments of this sort are, however, easily found in the literature in other areas of mathematical development such as proportional reasoning (e.g., Karplus, et al., 1983; Tourniaire & Pulos, 1985) as well as throughout cognitive development research.

$y = f(x) - 1$

$y = f(x - 1)$

$y = f(x) + 1$

$y = f(x + 1)$

$y = f(|x|)$

$y = |f(x)|$

$y = -f(x)$

$y = f(-x)$

$y = f(-|x|)$

$y = 2f(x)$

$y = f(2x)$

$y = \frac{1}{2}f(x)$

$y = f(\frac{1}{2}x)$

We had been interested in looking at how students interpreted transformations of these kinds. To that end, we tried to devise graphic images that students might choose among and talk about. Images like the following seemed to be quite appropriate.

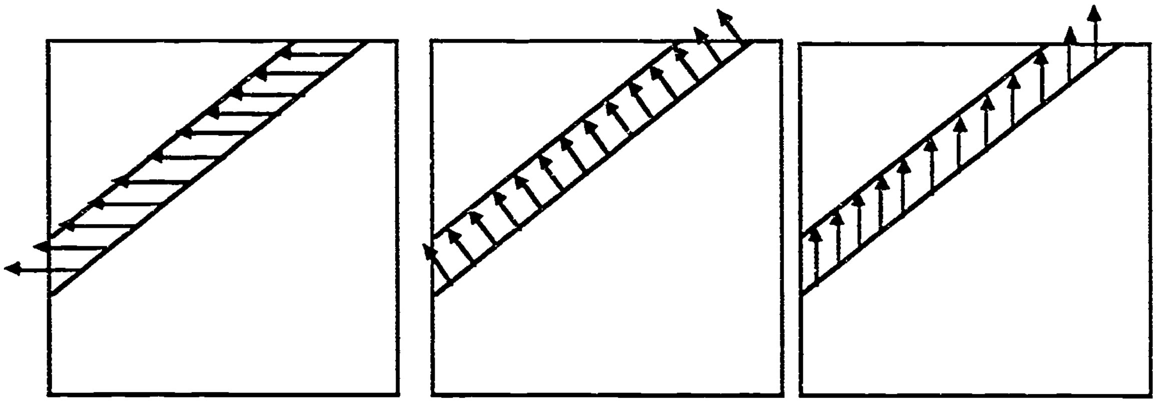


Figure 27

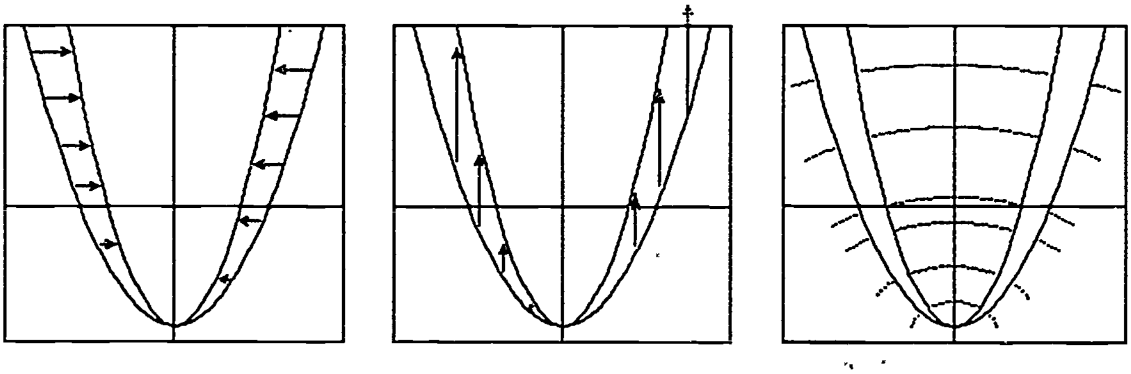


Figure 28

Although it may be easy enough for a student to express a personal preference about imagery, we as researchers cannot interpret these preferences meaningfully unless we can cast our questions in some unambiguous way. Remarkably, it turns out to be extraordinarily difficult to state questions about transformations unambiguously without, in effect, answering them. For a simple example, we would like students to see the addition of two functions as moving points up by the value of the added function. To assess what a student *does* see, we therefore might refer to the transformations on the linear function (figure 27) and ask which one best represents the movement one sees when b increases in the function $f(x) = mx + b$. A student can certainly be trained to give the "correct" answer, but it is reasonable enough to justify any of the movements shown in figure 27. To lock the student into a single interpretation of the question, we must specify something further like "for any *particular* value of x, how does the value of f change as b increases?" That is an important improvement of language for classroom purposes, but it gives the researcher too much influence over the student's response.

We had the same difficulty finding a sufficiently clear wording for the question about the parabolas and initially tried this:

> Each of the frames shows the graphs of the same two functions: $x^2 - 2$ is shown with a light line and $2x^2 - 2$ is shown with a darker line.
>
> Imagine an animated movie in which you see the graphs of these functions changing gradually from one to the other through a series of intermediate steps. Imagine further that you could watch individual points such as $(x_i, x_i^2 - 2)$ on one graph as they moved to their new locations at $(x_i, 2x_i^2 - 2)$. The arrows on each graph suggest three kinds of movement you might have expected to see in such a film. Which one of these figures do you think describes the movements of points from the function $x^2 - 2$ to the function $2x^2 - 2$?

Aside from the bulkiness, the problem with all such wordings is that they instruct the student to regard x as fixed. They thereby change the entire nature of the function: it ceases to be a function of x and becomes instead a function of the parameter.

In fact, the study of *families* of functions may lead to yet a third concept of function to add to the number-valued functions and the ordered-pair-valued functions discussed above: a graphical-object-valued function. This way of thinking about functions is reflected in our language when we make statements like "$ax^2 + 5$ ($a > 0$) generates an upward-opening parabola with its vertex 5 units up from the origin." It is certainly a natural enough way of thinking about functions, but it is not the *same* way as either of the prior two. This image suggests not $f(x)$ whose variable, x, is numeric and whose output is numeric, but $f_1(a,b,c)$, whose three variables a, b, and c are numeric and whose output is a particular graphical object, or an infinite set of numerical values, or the graphical representation of a subset of that infinite set, or, most generally, a function such as $f(x)$.

Done properly, this generalization of the concept of function may be exactly what we *want* to be teaching in the classroom, but, because it is likely to be new to many students, it may interfere with the interpretation of research results. Apart from ambiguities about what function we are discussing—e.g., a function f with the variable x or a function f_1 with the variables a, b, and c—we may be confounding the students' all too weak understanding of *variable*. As mentioned earlier, virtually all currently existing graphing software not only emphasizes the variability of the constants, but specifically isolates the user from the true variable by making it sweep automatically, continuously, and "invariably" from left to right without any user intervention.

Process versus object—point, slope, and function

Another way of viewing $f_1(a,b,c)$ is to regard it as a *function-valued* function. The concept of a function as a first class mathematical object which may be manipulated just as a number might be—that is, it may serve as a variable in another function or may be the output value of another function—is generally treated as an advanced one. Although this may not have to be the case—(perhaps a different route into mathematics than the one that we conventionally follow might make this notion more primitive and render some other concept that we now consider "easy" as the advanced idea)—most mathematics teaching except at a very advanced level treats functions as procedures and not as objects. Students naturally come to regard functions as a recipe specifying the steps to perform on some number (or, rarely, numbers) to get a numeric answer (Kaput, 1987). This is a standard preliminary step in mathematical learning: the procedure of counting develops into numbers; the operation of taking a part of gives way to fractions; students learn to differentiatie before they acquire a sense that the differential operator and its input might be regarded as a state of a function, an object that can be further manipulated (see Kaput, in press; Greeno, 1983).

This is akin to the process view of the point mentioned earlier, and like students' typical treatment of slope as a run-rise maneuver or a rise-over-run computation rather than as an attribute of a curve at a point.

There are consequences to the preference of procedural status over object status. For one thing, the arithmetic of vectors can remain very confusing until students stop viewing points as the end state of actions begun at a fixed origin and start viewing them as first class objects in their own right. In the same way, instantaneous velocity and slope at a point on the graph can be quite mystifying concepts when they are thought of only in terms of recipes that cannot be followed arithmetically in the world of infinitesimals.

If first-class object status is beneficial for a numerically defined ordered pair, what about an algebraically defined ordered pair such as $(x, x^2\text{-}3x\text{+}1)$, or the set of numerical ordered pairs that arises from such an algebraic definition? That is, would it not be equally beneficial for a student to develop a sense of functions that treats them as first-class objects? In a world of paper-and-pencil pedagogy, such a notion is nearly unthinkable until an advanced level, but it appears that with computer-aided experimentation with functions and their graphs, it may be particularly natural to regard functions in this "advanced" way almost from the outset, or perhaps accelerate the consolidation of the two views.

On the other hand, as observed earlier, students sometimes seem to lose whatever sense they may have had of the individual points along a line when they draw the line by the slope-intercept method. This might argue against a pedagogy that emphasizes the gestalt that comes with the graphical-object-valued view of a function. Alternatively, experimenting with the representation—finding points on the curves—may clarify what now remains obscure. As with each of the other issues raised, the matter is not yet researched.

How do points relate to each other?

The "mapping concept" of function suggests discrete arrows from each point in the domain set to each point in the range set. This image or some close variation on this image is useful in helping students to begin with the domain set each time and thus favor vertical displacements in transformations of functions in which the common views include horizontal displacements or displacements parallel to or normal to the curve.

But mapping is not the only way that functions are presented.

When we graph physical phenomena—e.g., objects rolling down hills, containers being filled or drained, temperature changing over the course of a day, trajectories of projectiles—we impose notions of causality and uni-directional time on them. These notions influence what transformations on the graphed relationship make sense to us. While we may reasonably ask how $f(x)$ behaves as x decreases in a functional mapping, the question becomes strange if x represents time: How does a car's speed behave as time rolls backwards? Similarly, we may reasonably speak of the functions f and g as inverses if $g(f(x))=x$, but if x represents time, the *meaning* of g becomes obscure or peculiar: how *do* fluctuations in temperature affect the passage of time?

What literature there is on students' interpretation of graphs has concentrated on experientially based meaningful situations. The clear advantage of this focus is that it recruits students' common sense, intuition, and reality-checking strategies. A disadvantage is that experientially based situations, in invoking causality and time-flow, exclude a large class of graphs that may require other kinds of thinking.

It is possible that the presence or absence of real-world referential meaning for a graph will matter in a student's interpretation of scale issues. Students may be more heavily invested in how a graph looks when the graph represents a familiar physical phenomenon than when it represents an arbitrary mapping.

It is also interesting to consider those contexts in which even sophisticated mathematical thinking tends to impose causality and uni-directional time on abstract graphs of function. For example, although we comfortably speak of approaching limits from either side, we also use terms like "increasing" and "decreasing" which are unambiguous only if we know the direction of movement along the x axis (Kaput, 1979).

Is there a difference between the way students understand graphs of functions as relationships with no temporal referent and the way they understand graphs of temporal processes? The question is interesting in its own right, but it is of especial importance to software developers. Point by point graphing by hand can proceed in arbitrary ways, but the conventional computer presentation of graphs sweeps smoothly from left to right, potentially emphasizing a temporal image of the function. Alternative presentations are possible and are, for some purposes, very clarifying.

Domain and range

Residues of earlier mathematical developmental stages

It is not uncommon for students to graph $f(x) = \frac{5}{x - 1}$ as shown in figure 29. Remarkably, even those who have already correctly drawn in a dotted line for the asymptote can make this mistake. Some even handle the apparent conflict by indicating the undefined value as a hole in the graph. The error appears to derive from at least two sources: students' predisposition to work with integers, and a deep conviction in continuity or, at least, connectedness (without necessarily a similarly deep understanding of these ideas). Lamentably, many graphing packages *also* cross the discontinuity with a connecting line, though not for failure to examine non-integer values for x!

In a similar way, avoidance of non-integers can cause students to flatten out the vertex of a parabola and miss other essential features of functions (see also Bell, et al., 1984).

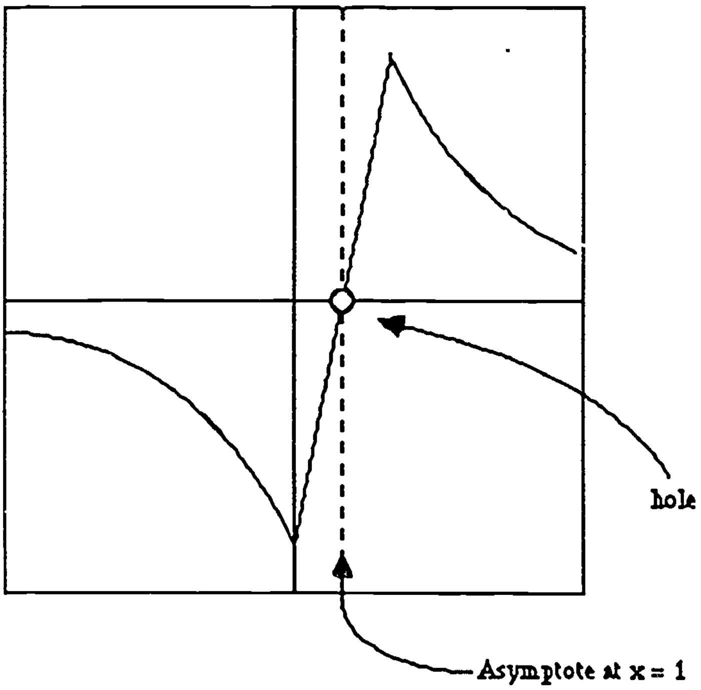


Figure 29

Students' further predisposition to restrict their work to *positive* integers alone can lead them into other kinds of trouble, e.g., the common conclusion that the graph of a linear function moves to the left as the constant term increases.

The illusion of finitude

When students look only at the symbolic representation of a function such as a second order polynomial they often easily recognize that the domain is unlimited: any value may be plugged in for x and a value for y may be computed. On the other hand, their visual impression of the graph of the same function often takes precedence over symbolic analysis and leads students to reason (e.g., in performing experiments with scale) as if the domain is bounded somewhere within the extremes of the domain depicted in the graph. In a graph such as this one (figure 30), x "looks like" it will never grow beyond roughly ± 10.

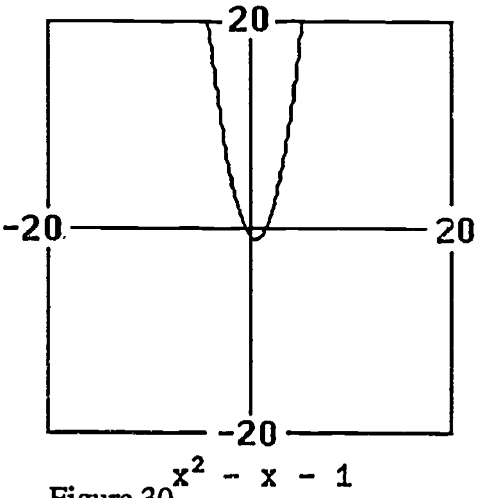


Figure 30

Even though the computer can provide easy control of scale which is essentially fixed on other media (changeable only by redrawing the graph at great effort), some of the comments of students and teachers with whom we have worked suggest that, at least for them, this sense of the limitedness of the domain may be more pronounced on the computer screen than on paper or the blackboard. Perhaps the operation of adding extra sheets of paper or waving one's hand with a flourish beyond the limits of the blackboard feels more familiar than somehow building onto the rigid computer screen.

Also, scale change alone may not be convincing enough. If, in addition, a student could drag the image one way or the other, perhaps the extent of the curve would be less obscured. With such control over scrolling, one could pull *along* the curve of the parabola and see that, though the y-position must change faster than the x-position, there is nothing to limit the x-position.

The finite and the infinite

To interpret the effects of scale change correctly, students must come to some reasonable understanding of infinite extent, and the meanings of arbitrarily large and arbitrarily small. One cannot, for example, use a zooming-in technique to get a better view of the hole in the function $f(x) = \frac{x^2 - 1}{x - 1}$. It remains invisible. Nor can one pull far enough away from a parabola to see the whole thing.

Robert Davis (personal communication) tells a story of a student whom he challenged with the notion that perhaps $\sqrt{2}$ did not exist. She martialled a graphic argument as proof that it did. Drawing a graph of $f(x) = x^2$, she drew a horizontal line from 2 on the y-axis to the parabola and then dropped a perpendicular from that point down to the x-axis. There, she claimed, was the location of $\sqrt{2}$ on the x axis. It was excellent logic to locate the value in that way *if it exists*, but her argument rests on the *assumption* that the value exists: that there is no hole in the parabola at that point, and that the x-axis is connected. It is interesting to compare this student's expectation with the difficulty encountered by the students who were trying to solve for x in two simultaneous linear equations plotted with the slope-intercept method. Those students *lacked* the sense that a point would be where they needed it; Davis's student implicitly assumed the existence of the point she needed.

Even some very familiar notions must be reexamined. For example, in explaining one's everyday experience with scale change, we referred earlier to the preservation of angle but non-preservation of distance under zoom-in/zoom-out maneuvers. The description applies well enough to any real view we may take out of a real window, but implicitly excludes the infinite. We cannot sensibly discuss which line is "longer": $f(x)=x$ or $g(x)=2x$.

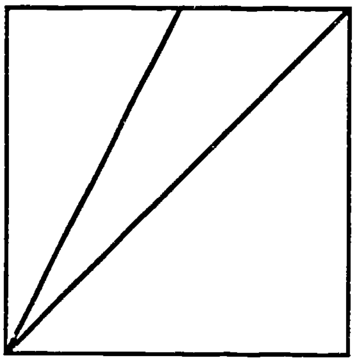


Figure 31

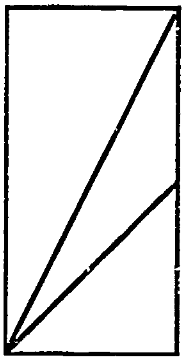


Figure 32

Segments may be compared as they appear on various graphs: on a square graph (figure 31), the less steep f appears longer; on a tall graph (figure 32), the more steep g appears longer. But the "length" of the whole line is not sensibly defined. Even equivalent number of points cannot be martialled as an argument because that argument applies equally well to the segments, which are certainly different in length.

In a pedagogy that makes regular use of translations between graphic and symbolic representations of function, we cannot avoid dealing with issues such as these, and yet notions of infinity and the continuous nature of the real line are totally foreign to beginning algebra students. Research needs to be done to find appropriate ways of dealing with these issues, and appropriate times to do so.

Issues particularly related to the computer

The advent of so much computer software leads us to ask particularly about issues which are related to the user interface of the software and the structure of the information provided by it. Constraints of the screen, types of support features built in, and modes of graphic presentation create their own environments. So do demands made on student involvement, the speed at which graphs are constructed, and necessary scaling choices. How do these factors affect the way students learn and investigate and their perceptions of the relationship between algebraic rule and graph? And what new tools do we need to investigate the effect of graphing software on students?

Features of the screen

Any physical graph, whether drawn by hand or by a computer, differs from the abstract mathematical ideal in that its "points" are not points and its "lines" are not lines.

The distinction between a point and a dot is mathematically important and is part of a larger idea that incorporates the inherent inaccuracy of measurement, the meaning of "significance" in statistics, the concept of "significant digits," and even the issues of infinity and continuity just discussed. The thickness of the blobs and swaths we casually refer to as points and lines creates no confusion for the sophisticated graph user who recognizes that the only mathematically important feature of these drawings is their positional relationship to the reference frame (the Cartesian plane) on which they appear. In effect, the experienced grapher knows what to ignore. The novice—the one for whom an *educational* graphing utility is built—does not. Scale is the only attribute of a graph that raises or lowers the significance of the distinction between a point and a dot. Because computer graphing makes it possible to play with scale—and because the computer's dots are regular and rectangular, making them easy to analyze—the computer is the ideal medium with which to tackle this important distinction head on.

A computer represents points as lighted pixels; curvature becomes a set of rectangular approximations. When a smooth curve occupies an entire high resolution screen, the illusion of a continuous curve is strong. However the same curve displayed in only a small part of the screen may no longer seem smooth, but rather like a set of boxes glued together at the corners. Furthermore, pixels usually aren't square, creating a whole new set of scaling issues and making some continuous graphs appear as discrete glued-together rectangles standing on end. Some students seem to think of the same graph at different magnifications as really being different graphs. What is the consequence of such a belief? The identifiable size of a pixel makes the hole in a graph, in $f(x) = \frac{x^2 - 1}{x - 1}$, for example, seem to have size, too. This may account, in part, for some students' apparent expectation that they can see the hole better if they magnify it sufficiently.

The pixel problem also affects a student's investigation of points of relative maxima or minima. If one uses a zoom-like feature on the computer to examine such a point at extreme magnification, the point appears not to exist. Instead, the computer draws what appears to be a horizontal line segment. Will students with lots of computer graphing experience think that the graphs of all parabolas have flat vertices? Or can this become an opportunity for entering the world of calculus through visual experience and intuitive argument?

Far more seriously, at high compression, pixellation may cause a local extremum to be missed altogether, generating a *vastly* misleading image. For example, figure 33 shows a parabola with a maximum around 250. Figure 34 shows the same parabola at a different scale. In figure 34, the vertex appears to be somewhere around -150. In fact, what has happened is that the vertex in figure 33 is not shown at all. The curve has been squashed horizontally to such an extent that all of the curve in the approximate neighborhood of $-5 \leq x \leq 10$ falls *between* adjacent pixels and therefore the computer is not able to represent it at all. Although clever mathematical design can reduce the frequency of problems like this, no amount of sophisticated software design can fully eliminate it; it is inherent in any system that represents points in a visible (and therefore finite-sized) way.

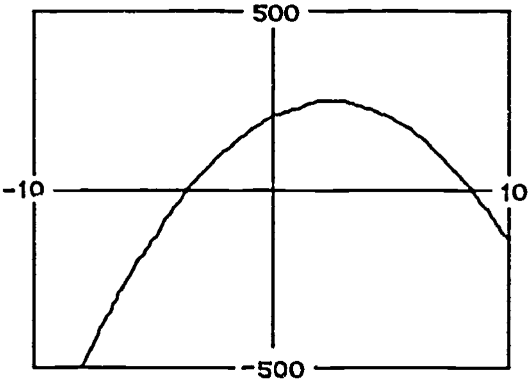


Figure 33

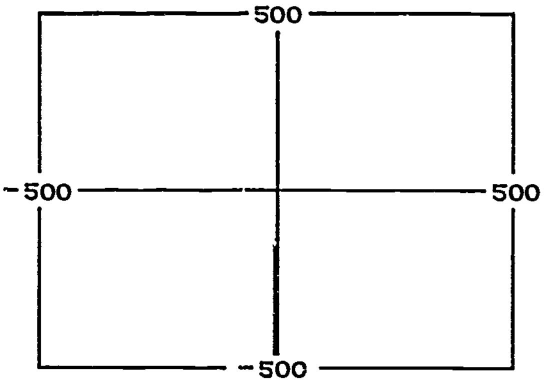


Figure 34

Whether pixel approximation becomes a valuable vehicle for important mathematical learning or, alternatively, a source of confusion and misunderstanding may depend on some screen and software design considerations but probably depends mostly on how the graphing software is used by and with students. Research is needed. How can software deal with the problem of magnification and pixel approximation? How can a teacher deal with the real issues of approximation?

New curricular choices

Pre-computer graphing was so tedious that teachers tended to allow students to move rapidly from the process of locating a few points to constructing a continuous graph. Sketches tended to be approximate and visual connections eccentric. However, the computer's speed and accuracy now makes it feasible for the first time to ask the student to *use* the graphs of functions to represent their rules. Through the medium of the computer, the student can manipulate either the symbolic or the graphical representation of functions, see the manipulations reflected in the other representation, and formulate and test hypotheses about the structure of the mathematics behind the displays.

What this means is that there are new choices not only in method, but in the content of the mathematics class. For one thing, as suggested earlier, it becomes possible to study families of functions in a way that was not possible before. Having the choice obligates us to make it. Research must be done to determine which way to choose.

Student involvement

Without a computer, the student must be involved at every level of producing a graph before experiments can be performed on it or with it. With a computer, most of the work of constructing the graph—including not just the point plotting, but determinations of scale and the identification of special features—can be done automatically. What is the impact of this on the student? Should software require students to do something significant to plot a graph or a point on a graph? If so, what should this "something significant" be?

Factors of the presentation

Usually, a computer-drawn graph is produced quickly and has the sense of being a single, global object. This already raises some questions.

1 Two features of computer graphing emphasize the gestalt and deemphasize the individual points that make up the visual whole: students calculate no individual points themselves; high speed graphing shows no point-by-point growth. Teachers in our research group have expressed opposing expectations about the effects of speed. Slow speed may foster a mapping image by leading students to see the curve as a collection of individual points. On the other hand, great speed may encourage more exploration and may, by showing the gestalt, help develop a notion of function as object. Teachers have expressed a similar ambivalence about the reduced computation. Again, it helps focus attention on the overall behavior of the function, but teachers fear that students may need some replacement experience to help them connect the visual gestalt with the mapping of individual points on the number line that it represents. Experience with classes is needed in order to learn which skills and understandings improve and which weaken.

2 In many pieces of software, the student makes parameter choices and the computer presents the graphic output without further student involvement. There are many alternatives to this kind of presentation. One might allow the student to mouse a point or a continuous stretch of points on the domain and have the graph appear only at those points that were selected. Giving the student control over the order in which calculations are performed and points are plotted may be a suitable substitute for the hand calculation and plotting that the student misses in conventional computer graphing software. Allowing points to be plotted "out of order" also returns one to the image of function-as-mapping rather than the function-as-modeling image that is so strong in left-to-right sweeps. Further, by suppressing the ability to get continuous stretches, the student's attention may be drawn back to questions of how to locate local features of a graph—picking a point *between* other points to see how the graph behaves in that neighborhood. How would this kind of presentation affect a student's perception of the nature of points on a graph?

3 We can also imagine that a piece of software like that described in (2) could provide coordinates any time the mouse was clicked, forcing an association between point and pair. Would the effect of this association be positive?

Scale and scale information

If software is to allow a user to experiment with a wide variety of function graphs, it must make scaling choices to make the graphs fit. Most allow the user some choice of scale as well: if the graph of a function seen at a distance is uninformative, one wants to zoom in to look more closely at a section of the graph. The designer of such software must not only decide what choices to offer, but must make choices as to the variety and style of information presented on the screen. Clearly, if scale changes have been made, the user must be provided with information about the scale. How should this be presented? A full grid, like graph paper? Axes with tick-marks? Unmarked axes with values marked only at the borders of the window? Dots at certain integer grid-points? All of these as options?

And what kinds of scale make sense to students? Students seem to prefer symmetric scales on their axes even when these obscure important features of the function they are viewing. For example, a pair of students tried several times to change the graph shown in figure 35 so that its axes would be symmetric. The resulting views (figure 36) were seriously deficient.

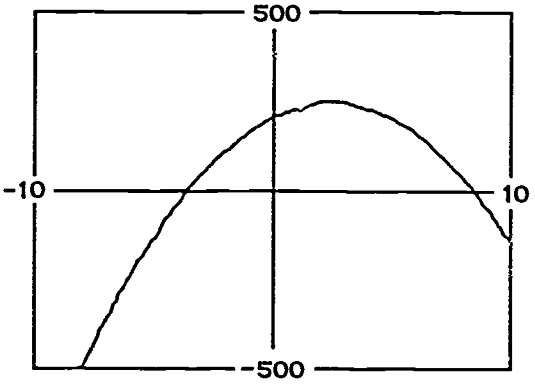


Figure 35

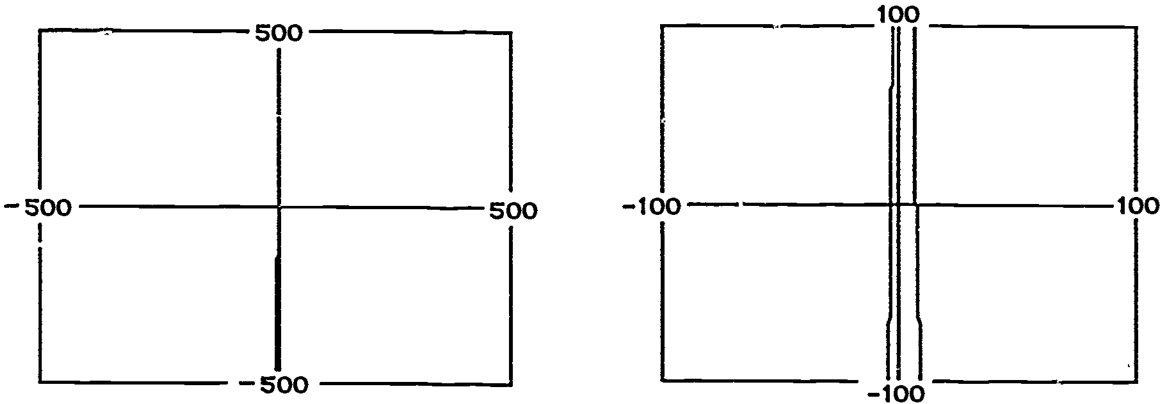


Figure 36

Independence of scale on the two axes is sometimes essential just to see the graph, but there is a mathematical significance to this independence as well. In a graph of velocity vs. time—and in general, for any function with referents that are measured in different units—symmetric scale has no meaning.

Is the general preference for symmetric scaling merely an artifact of students' first classroom experiences with graphing or are expectations about symmetry developed even prior to that, perhaps by experience viewing objects as they approach or recede into the distance? And how tenaciously do students hold to this limited view of scaling that is so counterproductive to their efforts to make the graph more visually accessible? What tools do students already have that can help them visualize the effects of scale changes, and what new tools do they need?

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Even our language for describing scale-change operations is weak. Suppose we renumber the x-axis without changing the size of our graph window: in our new scale, each unit of space represents a field that is ten times as large as it was in the old scale. We might think of that as *expanding* the scale by a factor of ten, either because we are expanding our view or even because the numbers have grown larger. On the other hand, we might think of it as a *compression*, because more territory fits into the same space: a comparison of the same function graphed on the two scales suggests compression along the x-axis in the new scale. We simply don't know which is the prevailing image—expansion or compression—nor do we have enough experience to inform us as to which is the more useful terminology to adopt, but the language must be consistent, either with respect to points or their labels.

The issue is really one of alternative perspectives and confusion about what is the object that moves and what is the referential system against which the object's movement is to be measured. We expand our view; the scene is compressed. Problems of this kind occur regularly in day-to-day experience with respect to time and position as well as to scale. What do students most naturally see as the manipulable object in a stretching action? Is it the function alone, as if it were drawn on a transparent rubber sheet and can be stretched or translated over a fixed axis system? Is it the axis system alone, as if the function were an independent, free-floating, real-world object and the axis system were not attached to it but could scan over it like the hair-line on a gun-sight? Or is it both, as if the entire graph, axes and all, were constructed on a rubber sheet? Or is it better yet to have an image in which both the axes and the appearance of the graph are fixed and it is the student who shifts attention from one region to another and approaches or backs away?

Distinguishing the "zoom" operation from other scale changes

The image of approaching and backing away brings up another issue that raises conflict with our intuition.

In the real-world, the horizontal and vertical *changes* in the dimensions of objects that approach and recede are equal: if the horizontal scope of view is doubled (that is, if the horizontal dimensions of any *object* in that view are halved), then the vertical view is also doubled. If one is tripled, the other is tripled. Even when total independence of x- and y-*scales* is accepted, our intuition about scale *change* is so tied to our experiences with changing distances in the real-world that when it is our intention to perform a zoom-like operation—one that simply gets "closer to" or "farther from" the function in some meaningful way by rescaling both axes—our almost automatic approach is to change both scales by the same factor, preserving the ratio of the two resulting scales.

Such an equal rate change might very reasonably, however, be treated as a special case. Consider the visual effect of scaling the units on both axes by a constant m. For $m = 1000$, this represents changing both the x- and y-scales by a factor of 1000 and moves, let us say, from a view that includes a radius of 10 around the origin to a view that includes a radius of 10,000 around the origin. If the function that has been graphed on the initial grid is linear, we are not shocked by the result. Distances shrink and angles are preserved just as we expect them to be. In particular, the apparent distance between the function and the origin diminishes, but the function retains its characteristic slope. Another way of saying this is that the two functions $ax + b$ and ax become more and more indistinguishable the farther out we zoom. In this case, the equal rate change of scale seems to be what we would want for a zoom.

The same rescaling operation, however, is often very unsatisfying when it is applied to a parabola or cubic; its effects do not meet our casual expectation. Zooming out far enough in this way leaves a parabola looking like a vertical ray (see, for example, figure 36) and leaves a cubic looking like a vertical line. Thus, this familiar conception of the zoom operation has different visual effects depending on the polynomial to which it is applied.

For polynomials, at least, a different conception of zoom might make more sense. Consider a zoom that is linked to the order of the polynomial. For a cubic, if the x-scale were changed by a factor of m, then the y-scale would be changed by a factor of m^3—for example, zooming out until the x-scale is changed by a factor of 10 would cause the y-scale to be changed by a factor of 1000. The farther "out" we move with such a zoom, the more indistinguishable the general cubic $ax^3 + bx^2 + cx + d$ becomes from ax^3. Similarly, if the y-scale changes by a factor of m^2 as the x-scale changes by a factor of m, then the quadratics $ax^2 + bx + cx$ and ax^2 would appear to converge on zooming out. Perhaps more to the point, the root functions—ax^3 and ax^2—would not appear to change at all under the order-linked zoom operation, just as the line ax does not appear to change at all under a conventional linear zoom.

The mathematical significance of this kind of order-linked zoom is so great that we may want to go out of our way not to support the already strong bias toward linear-space zoom.

And in the case of functions with referents, it is difficult to make consistent meaning out of *any* definition of zoom. What, for example, does it mean to "approach" the velocity-time plane? That is, what kind of measure—distance? time? pounds?—are we reducing as we "approach"? Although this may seem a tiny matter when all one seeks is a better view of the graph, the metaphors we use affect our understanding of the mathematics. In getting a better view of a velocity time graph, we want to change scale at our discretion, but we do not want *zoom.*

LESSONS TO BE DRAWN

There is no fitting summary for this work: not enough is yet known about the various phenomena we have noted to make a summary statement at this time. But there are some lessons that may be drawn now. Here are a few.

1 Students base comparisons of graphs on gestalts that they create by matching identifiable "special" points such as, in the case illustrated in figures 1-4, the vertex of the parabola. To help counter the attention that the vertex draws, it may be important for software to let students mark points whose transformations they wish to track. One such point might be, for example, the y-intercept.

2 We must be thoughtful in the language we use. The two parabolas in figure 2 *are* graphed at different "heights." The use of the word "height" to refer to the effect of the constant term is equally sensible but quite different. Language being what it is, it would be unnatural (and probably ineffective) to proscribe one of those uses, but the difference should be made clear by drawing explicit attention to it.

3 The perceptual illusion illustrated in figures 4, 20, and 21 suggests that students need to have experience with shifting figures around on the screen and watching the effects of those shifts. When such figures are produced by a *graphic* rather than symbolic manipulation—dragging the parabola vertically with a mouse or moving it stepwise with an arrow key—the illusion tends to vanish, as one is more apt to measure distances along the direction of movement than along a shortest path.

4 The *structure* of an algebraic expression may affect a student's interpretation of its graph, so multiple forms must be available for students to explore and translate among.

5 When multiple scales are used to represent the same graph, graphing windows should contain internal frames or other visual aids to help students recognize which portion of a distance view is being enlarged in a close-up view. See, in particular, figures 18 and 19 for examples of this kind of aid.

6 It is important that students have experience *controlling* scale. In fact, students need experience both with strictly metric controls (e.g., specifying the exact borders of the region of the plane they wish to examine), and with visual, primarily non-metric controls (e.g., stretching or shrinking an image to reveal or match some property without having to compute window border values in x and y). Default behaviors of scaling controls, e.g., a zoom function of some sort, should support richer rather than poorer notions of the scaling of space. In particular, we should not reinforce students' preferences for symmetry and linearity.

7 Thought must be given to the kinds of educational questions that we ask. Not only does rapid graphing make new questions possible, but it makes *some* new questions, e.g., ones that draw attention to the possible ways of misinterpreting the newfound graphic information, quite important.

8 Our students, even at early stages, cannot escape having to deal with notions of continuous functions and discrete points, infinity and infinitesimals, the invisibility of points, and other issues we tend to ignore until the calculus. We must consider appropriate ways of introducing these ideas much earlier than we typically do. The behaviors of some young students who become comfortable changing scales on graphs suggests an "intuitive calculus" long before the algebraic manipulations for formal calculus are present.

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III. RESEARCH INSTRUMENTS AND TRAINING MATERIALS FOR SECOND YEAR

OVERVIEW

Near the end of the first year's work, we began to focus our attention on planning for the second year of research. Two considerations led us to believe that our best strategy was *not* to develop a single line of interview questions and train research assistants to conduct reliably comparable interviews using the prepared questions.

1 The scale issues which are our focus interact complexly with many other issues. Until we know how students respond to our probes, it is difficult to predict how we will have to proceed in order to tease apart what failures are due to scale alone.

2 Those probes that we have (see, in particular, Appendix C) that *do* focus quite purely on scale, effectively eliminating the interaction problems reported above, remove scale issues so totally from the graphing activities that motivate our study that we feel we must perform other experiments as well. Put differently, our purpose is to study scale *in the context of graphs of functions*. We believe this to be an inherently complex domain. Although we must certainly develop and use probes that help reduce or control for the complexity (e.g., the Appendix C probes), we cannot ignore viewing the complex situation of interest itself.

Therefore, we developed statements of rationale for the interview as a whole, with a very broad set of examples of possible interview items.

To give researchers a "head start" on creating follow-up questions as needed in the course of an interview, we decided to create problem-based teaching materials for what we think will be the most important of the diverse paths that interviews might follow. These are explained in the next section and attached as apeendixes. Altogether, these provide more strategies and more variety than any single interview could use.

As practice for interviewing, research assistants will observe interviews and perhaps conduct some jointly with a senior member of the research team.

AN EXPLANATION OF THE PREPARED MATERIALS

Probes

During the first year, we created a set of interviewing guides. We call them probes. These tend to take the form of worksheets that present a situation and ask some questions about it. Most probes have been created around some hypothesis concerning confusions that students have regarding graphs and functions. Most can, therefore, serve either as the basis for an interview or as a basis for teaching intervention. In the course of our investigation we may use both methods.

Some probes make use of the computer while others do not.

Appendix A contains non-computer probes designed to find out if a student can interpret and create a variety of real-world graphs. Because, for some students, a real-world model may be a necessary precursor to abstract graphs, we felt these would help to clarify failures in tasks with the abstract probes.

Appendix B contains probes designed to explore the connections a student has made between a complete graph and the symbolic representation of the function that produced it.

Appendix C contains probes dealing with issues of shape and scale.

Appendix D contains several computer explorations. Some of them are self-explaining and students may work without intervention from the interviewer. The interviewer's role is in those cases is to encourage thinking aloud, tape record all responses, and ask questions where necessary to clarify a student's thoughts or procedure.

In the remainder of this section we shall discuss some aspects of the design and use of these probes.

Interpreting and creating graphs of real-world phenomena: Appendix A

The probes in this section are designed to assess (1) whether or not a student is familiar with graphs of real-world phenomena and their interpretation; (2) whether the student is capable of creating and interpreting such graphs correctly with particular reference to the appropriate use of points and lines to distinguish continuous from discrete relations. The probes are designed in worksheet form, but should be presented in a form appropriate to an interview situation, e.g., omitting phrases like "If you have answered 'yes' to #3, then...."

Abstract rules and their graphs: Appendix B

The probes in this section are also cast as worksheets. Most deal with traditional issues concerning the interpretation of graphs and the algebraic notation usually applied to them.

Issues of scale: Appendix C

Issues of scale rarely arise in elementary algebra graphing exercises for two reasons: (1) teachers and students select functions whose graphs fit on standard graph paper so scaling is not necessary; (2) texts customarily promote graphing using slope and y-intercept so early that introducing scale considerations would interfere with the student's acquisition of the concept of slope. By contrast, the use of graphing software to explore functions requires students to confront these issues.

In an attempt to separate scaling issues from other graphing skills, we begin our scale probes not with graphs but with a real-world model suggested by the word scale itself: the world of thermometer scales.

Scale 1

These probes contain questions the student can answer without the interviewer's intervention. The interviewer's primary role here is to encourage thinking aloud and to ask additional questions as necessary so that we have as clear a record as possible of what the student is thinking.

Scale 1a The idea here is to deal with scale cleanly in as simple a context as possible. The one dimensional issue of translating one thermometer scale into another seems both an appropriate model and one most students have some experience with. This probe involves only a change of scale, but no change of origin.

We are interested in what a student does when asked to diagram the Nory scale next to the C scale. The possibilities are many. Is 0° N opposite 0° C? Are the segments the same length? Are the divisions equal? When a student does something, the interviewer should be on the lookout for pitfalls which occur as a result of that action. For instance, suppose a student makes 1° N exactly equal in length to 10° C. Does this mean that the student thinks that 1° N = 10° C or is it a case of casual inaccuracy in drawing? And is this used in the process of conversion?

Once the scales are drawn, the student is asked to draw arrows in order to provide a visual scale correspondence. Does the student interpolate visually to perform calculations? Under what conditions does the student perform calculations from a rule? Can the student see that a rule can provide the same correspondence? Watch for students who show a clear preference for either the rule or the visual match. Does there seem to be a consequence in their interpretations or computations? Does a student maintain a visual preference even when the information is clearly not precise? Where students behave very differently, it may be worth asking their teacher for other information about them.

Scale 1b This probe introduces a translation (a change of origin with no change of scale) and some further abstraction. Does the student shift the K scale visually so that the 0 point corresponds with the N scale's 3? What effect does a non-shift have on a student's performance? Do shifters do better on later questions? Does the student move between the K and N scales to obtain information about the C scale? Does the student prefer to use calculations obtained from the previous sheet?

Scale 1c-1f These represent a different model for an interview. They consist of pictures and suggested questions (listed on the sheet) to asked by the interviewer. It is not possible for a student to involve herself in these probes independently. The student's graphing sheets should not contain the interviewer's questions.

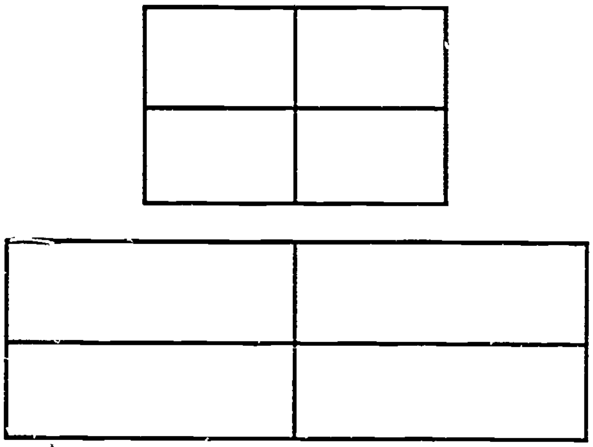
The idea here is to make a transition to the two dimensional world of graphs. There are unexpected dividends here. The information provided by the matching arrows between parallel scales is much more efficiently depicted by a point when the scales are perpendicular. It is fundamental to graphing skills that a student realize this. The probe gives the interviewer an opportunity to assess and teach if necessary. In fact many fundamental questions about graphing can be asked. Does a student see what information a point conveys? To what extent is it possible to move in both directions C to N and N to C? Does a student favor one direction? Does a student naturally answer a question by interpolating a point on the line once three points have been graphed? Does a student use the graphs to answer questions about negative numbers? Does a student see a new line as a new thermometer pairing and is the student able to use it? Is the question about 1 N = ? C facilitated by the graph?

Scale 2

Scales 2a-2d are left without fixed questions so that they can be used as support materials for the interviewer (e.g., to record responses to other probes). The format attempts to get at the problem of stretches and shrinks as they are generally depicted on a computer screen. Each sheet contains a pair of windows with axes in them.

Scale 2a	The windows have the same height and different lengths.
Scale 2b	The windows have the same length and different heights.
Scale 2c	The windows have different lengths and different heights.
Scale 2d	The windows are identical.

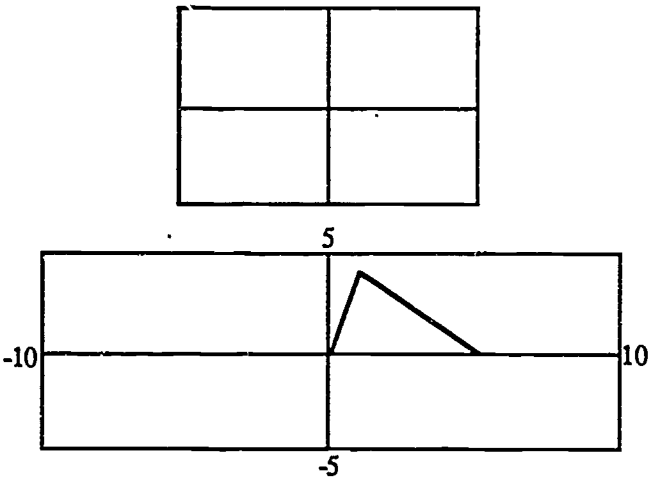
Scale 2a looks like this.



In addition, there are two classes of questions that these forms suggest. Can the student process coordinate information to determine the shape of an object (e.g., a triangle) which is transformed from one system to another? Can the student see how transformational information can be processed to yield coordinate information?

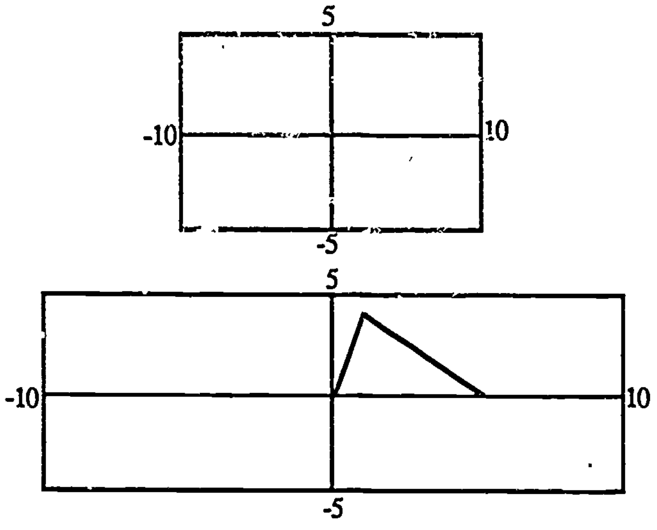
One strategy for exploring scale independent of function might be the following:

Interviewer places coordinates on the axes in the large frame, and then sketches a simple figure, e.g., a triangle in that frame.



The interviewer then asks the student to sketch the same figure in the smaller window. Does the student automatically make a congruent copy? Does the student assume from the picture that there is a compression of the system and therefore change the shape of the triangle? Or does the student realize that more information about scale is needed to do the job?

The interviewer then places coordinates on the small window and asks S to perform the same task, using a different sheet if S has already drawn a triangle. Do the coordinates affect the drawing? Ask S to approximate the coordinates of the vertex of the triangle. Does S realize that the coordinates of the small-window image should be the same?

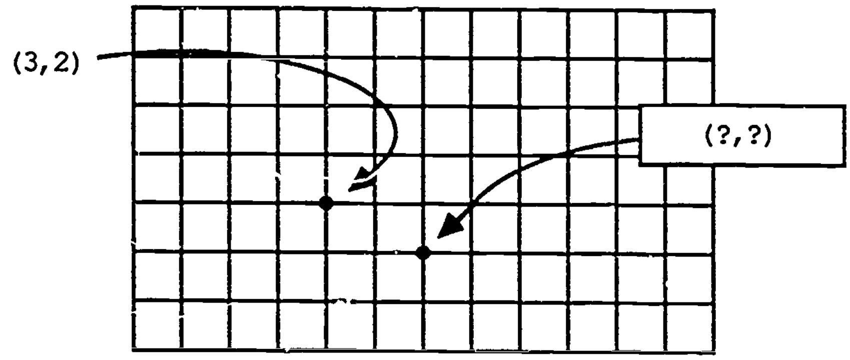


At this point, the interviewer might want to explore the results of a variety of scales, say -5 to 5 or -20 to 20. He might then want to go on to draw the triangle in the small window and reverse the process, asking essentially the same questions. Is it easier for some students to go from large to small and others from small to large? What happens when the numbers don't match the visual image, that is when the larger window has the smaller coordinate limitation?

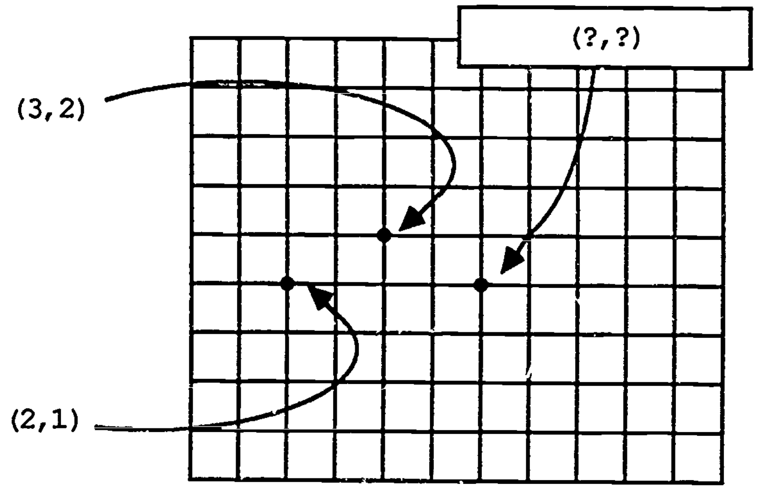
The same exploration can be performed with constant horizontal and variable vertical dimension, with changes in both dimensions, and finally with the sheets on which the windows are the same size. Much graphing software handles scale by fixing the window and giving size information, and it is important that students be able to deal with the scale-impact of fixed windows and changing coordinate systems.

It will be important to identify skill level at each stage. Perhaps students who can handle questions involving coordinates which conflict with the relative sizes of the windows can automatically handle the fixed-size pairs. Or the converse may be true.

The selection of geometric figures consisting of more than one point was motivated by our desire to make the problem more concrete than the exploration of the coordinates of a single point, but it may be a drawback if the complexity of multiple points connected by lines is a confounding irrelevancy. Similar questions may be asked about the location of single points on a graph. Another version of the single-point question works like this:



Does the student see this task as having no solution? One? Many? Does the student know what information is missing? How does the student approach the problem?



Does the student see this version as more or less complicated? Does he or she recognize that all needed information is present? What other information does the student request?

Scale 3-6

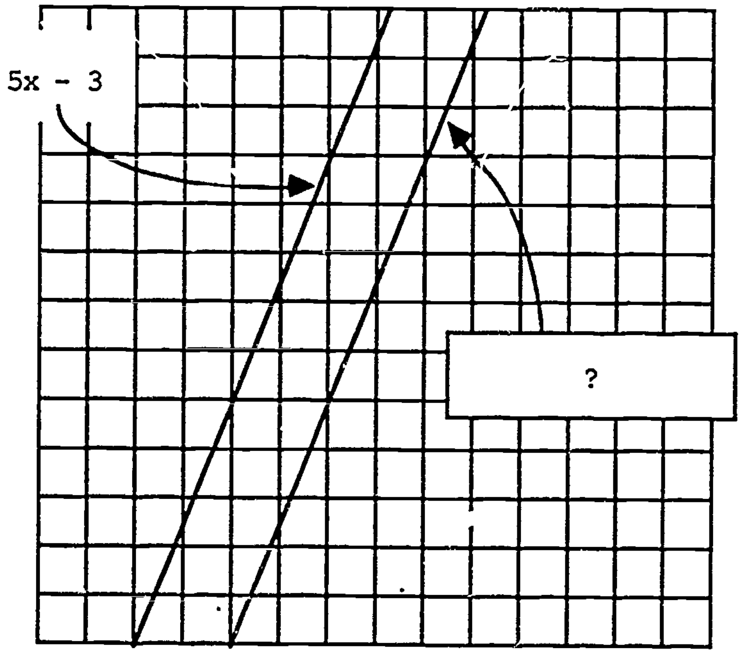
These sheets explore the same concepts as sheets 2a-2d, but in the context of the graphs of functions. Previously, the object (the triangle in our example) had easily identifiable points. The figures the interviewer sketches in this section will be graphs of functions, often without such clearly discernable features. We think now that it is better that the rules not be identified, although the interviewers may decide differently. It is also not clear what progression to follow. The conventional algebraic order with linear forms presented before higher order polynomials may be unfairly devoid of features. On the other hand, they may make for fewer sketching issues. This will have to be seen later. Examples should include functions that pass through the origin and ones that do not. We think it is satisfactory for the interviewer to draw the curve, perhaps introducing the problem in a manner like this:

> Have you sketched the graphs of algebraic rules or equations before?
> Give me an example of a graph you've seen before. (Have the student sketch something)
> Suppose I put down these numbers on the window here. What does that tell you about the graph?
> Ok, can you sketch the same graph in this little window?
> Suppose I put these coordinates here. Can you sketch it now?
> Why did you make it go this high here?

This questioning game can be played with each of the functions if the intervier keeps in mind that she is looking for patterns of visual interference with numerical information. In particular, are the patterns with function-graphs the same as those she has observed for the object-graphs in the preceding sheets?

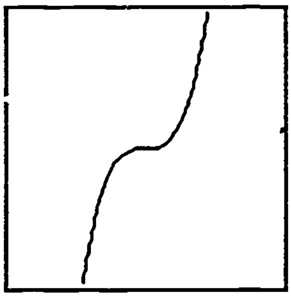
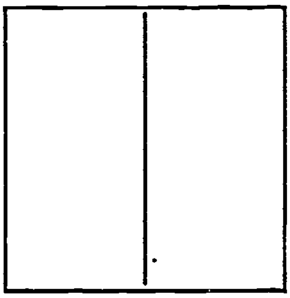
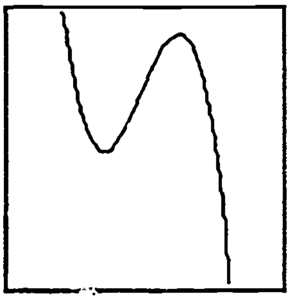
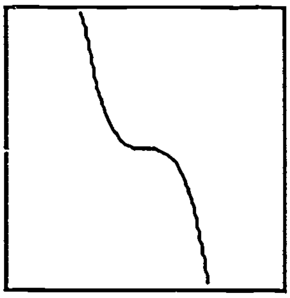
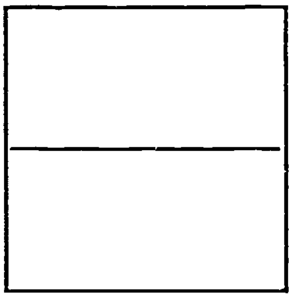
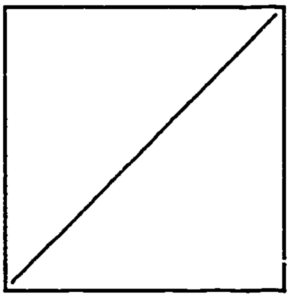
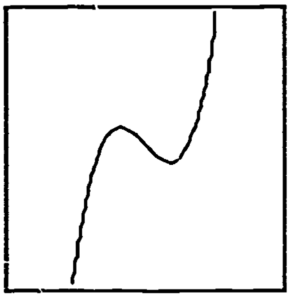
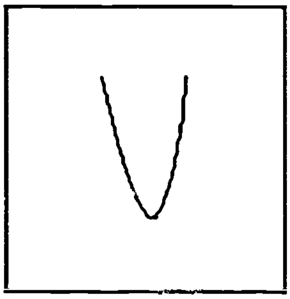
The interviewer must also explore with appropriate students the connections which the student has made between the graph of a line and the symbolic representation of the same function. The interviewer might draw a line in a coordinatized window and ask about the slope or intercepts. Then, using another sheet of the same type with recoordinatized windows, ask the student to draw the same graph. Does the student preserve the slope visually, or account for the change in scale? Where is the *y*-intercept located? How is the process affected by going from large numbers to small or from small to large?

Again, the interviewer may turn the problem around in the following way:



Does the student recognize what information is missing? Does the student recognize what information is *not* missing?

The interviewer may also present tasks, on or off the computer, that ask a student which of a set of figures might represent the same function drawn on different scales. The essential information will consist not in what the student picks, but in what explanations for those choices the student offers.

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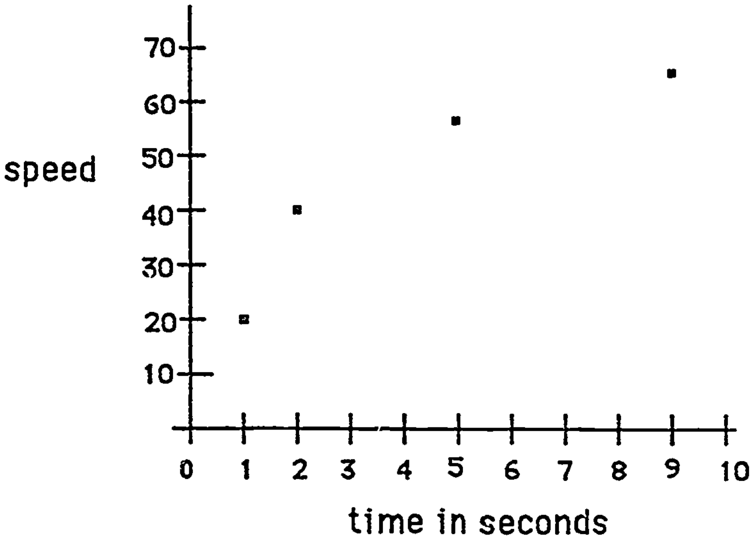
V. APPENDIXES

APPENDIX A

MODELING

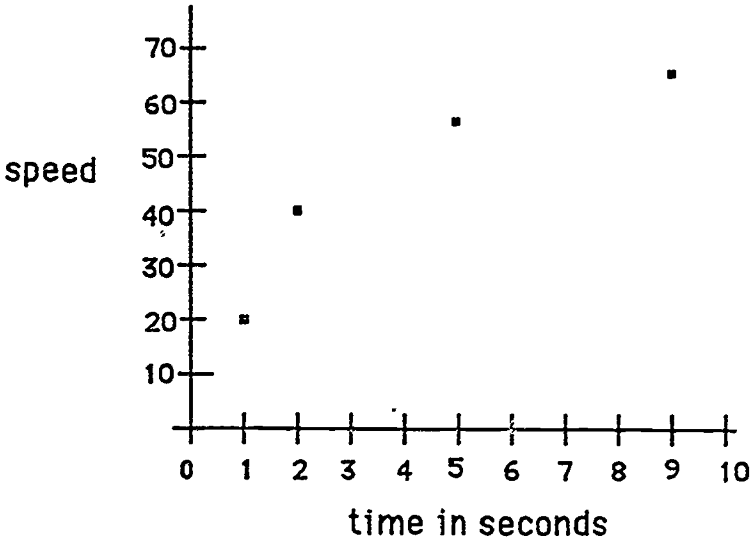
The diagram at the right was constructed from a table of information that was recorded after a test driver tested a new car.

The horizontal axis represents elapsed time in seconds. The vertical axis represents speed in miles per hour.



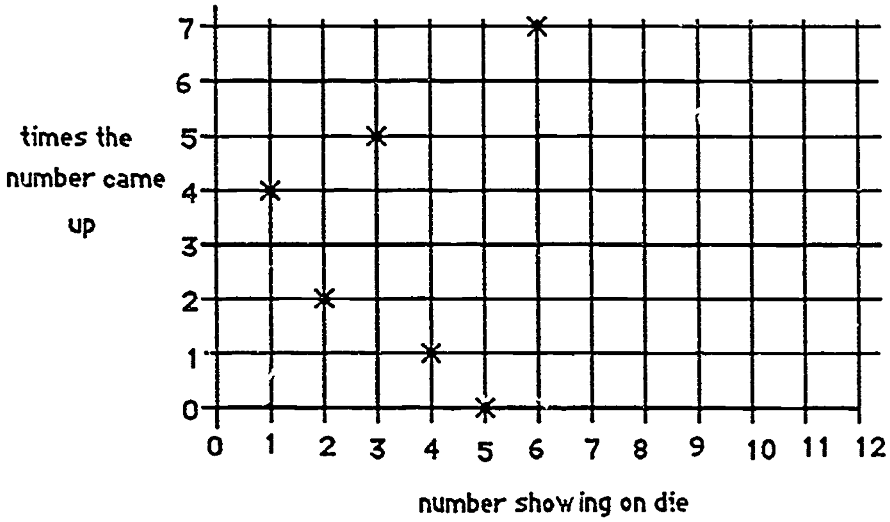
1. Circle one of the points on the graph. What information does this point give you about the test?
2. The four points on the diagram were created from a much larger table of information. Locate a 5th point that might have been created from the table and mark it with an x.
3. Assume that the actual table from the test consisted of many entries. Would it make sense to connect all these points to make a graph? ______ Explain your reasoning in a few words.

4. If you answered *yes* to question 3, draw such a graph here.



Interviewer Date Student

The diagram below was constructed after one die was rolled a number of times. The horizontal axis depicts the numbers on the face of the die. The vertical axis depicts the number of times a number came up.



1. How many times was a 3 rolled?

2. How many times was a 5 rolled?

3. Which number came up 6 times?

4. Which number came up once?

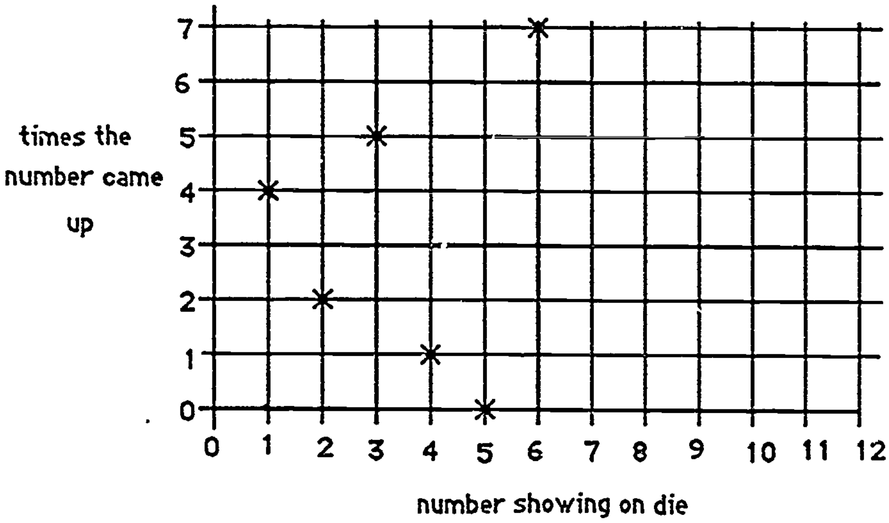
5. The graph above consists of just 6 points. Where might a 7th point be? ________

Could one point lie between two points that already exist?______

If so, mark a possiblity on the graph and label it with x.

Interviewer Date Student

6. Some people have connected the points on this graph. If this makes sense, do so below. If it doesn't, explain why it doesn't.



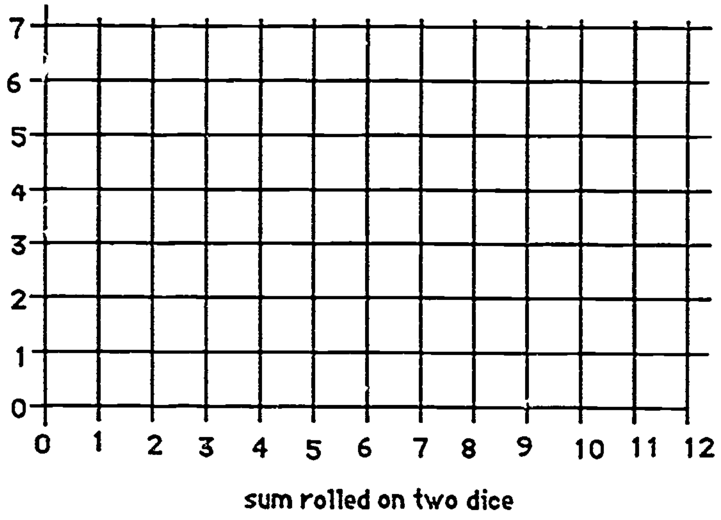
7. How many times was the die rolled? If there is no answer, say so. If you think there is an answer but are not sure how to get it, put down a question-mark. (?)

Interviewer Date Student

Two dice were rolled 12 times. The following sums came up:

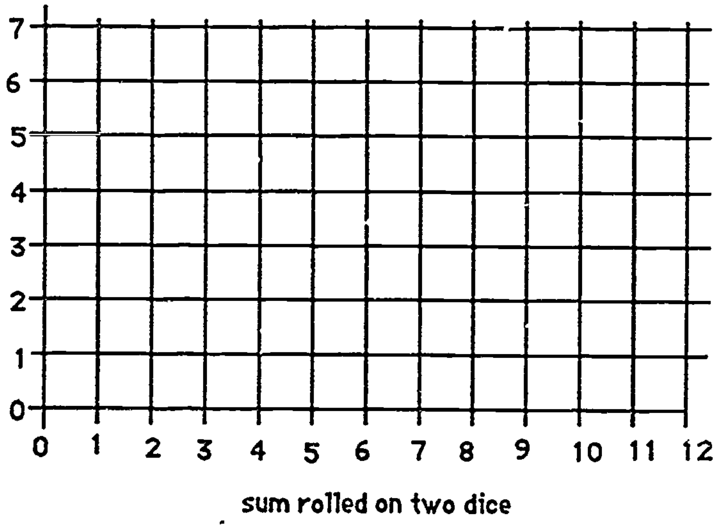
2 - one time
3 - one time
5 - two times
6 - one time
7 - three times
8 - four times.

1. Graph the data on the grid below.



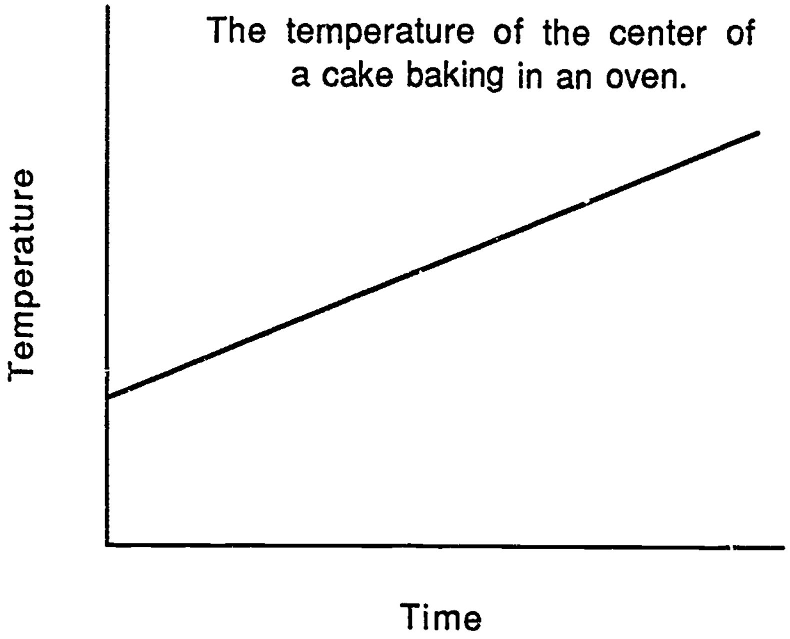
2. The dice were rolled three more times. The following sums came up:
   9 - two times
   6 - one time

   Sketch the graph which represents all 15 rolls. (Extend the grid if necessary, providing any labels required.)



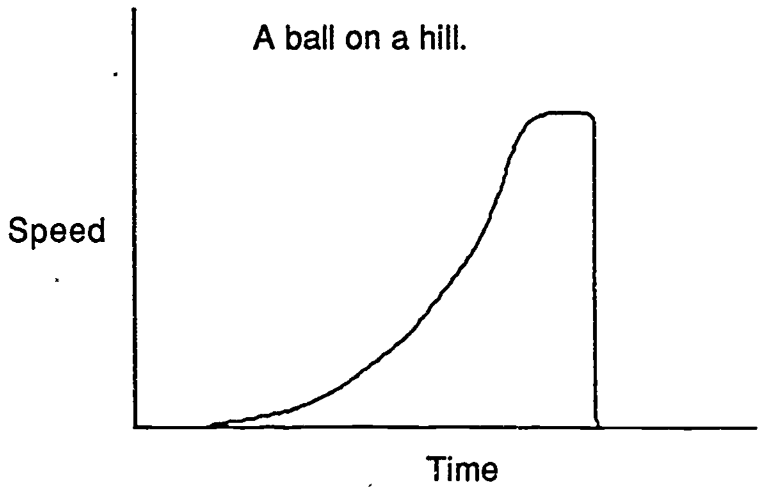
Interviewer Date Student

Carefully examine the graph. Then describe in words what the graph says.



Interviewer Date Student

Carefully examine the graph. Then describe in words what the graph says.



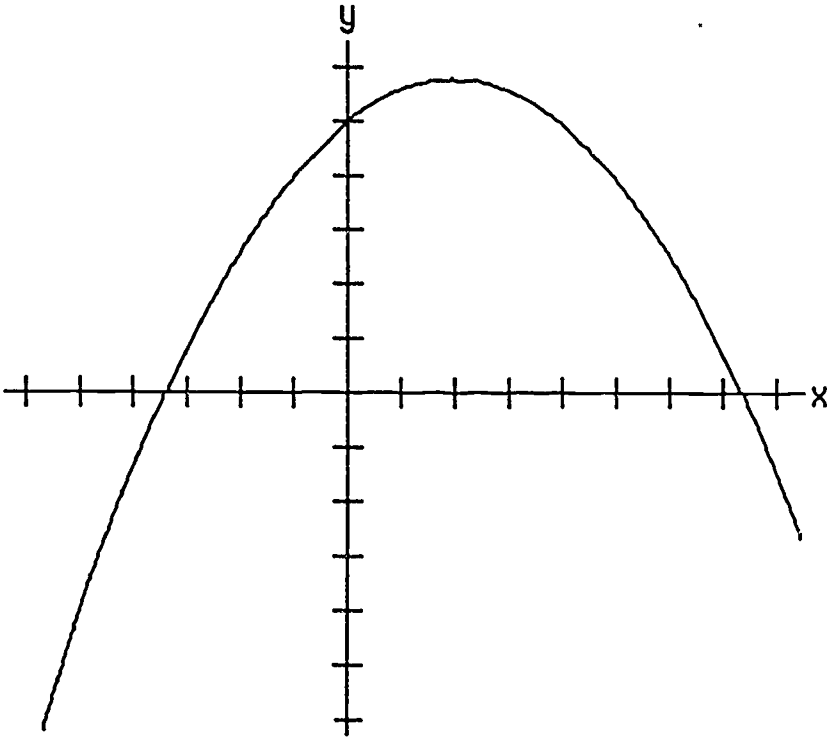
Interviewer Date Student

APPENDIX B

MAPPING

On the grid below, each mark on the x-axis is 1 unit; each mark on the y-axis is 1 unit.

The graph sketched is a curve called a parabola.



Give the approximate coordinates of the following points on the graph:

1 The point(s) where the parabola is the highest.

2 The point(s) where the parabola crosses the y-axis.

3 The point(s) where the parabola crosses the x-axis.

4 Is the point (4, -1) close to being on the parabola?

5 Is the point (-1, 4) close to being on the parabola?

6 Find the y-value(s) of any point(s) on the curve whose x-value is 3.

7 Find the x-value(s) of any point(s) on the curve whose y-value is -2.

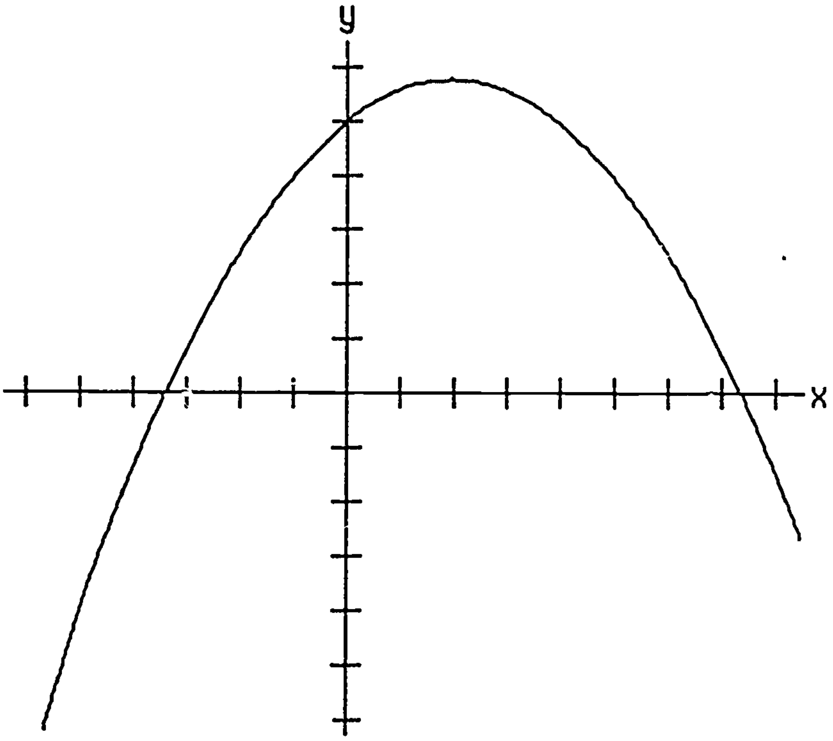
8 Find the x-value(s) of any point(s) on the curve whose y-value is 6.

9 Find the x-value(s) of any point(s) on the curve whose y-value is -4.

10 Find the y-value(s) of any point(s) on the curve whose x-value is 9.

Interviewer Date Student

On the grid below, *each mark on the x-axis is 2 units; each mark on the y-axis is 5 units.*



The graph sketched is a curve called a parabola.

→ **Reread the description at the top of the page.** ←
Then give the approximate coordinates of the following points on the graph:

1 The point(s) where the parabola is the highest.

2 The point(s) where the parabola crosses the y-axis.

3 The point(s) where the parabola crosses the x-axis.

4 Is the point (4, -1) close to being on the parabola?

5 Is the point (-1, 4) close to being on the parabola?

6 Find the y-value(s) of any point(s) on the curve whose x-value is 3.

7 Find the x-value(s) of any point(s) on the curve whose y-value is -2.

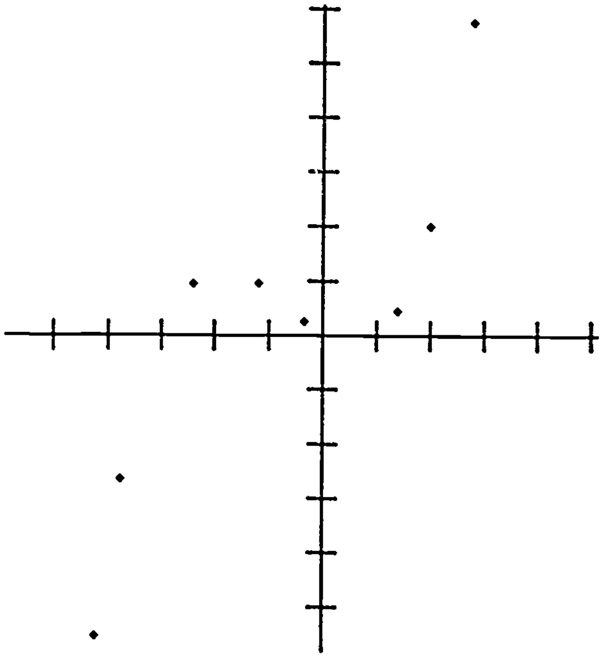
8 Find the x-value(s) of any point(s) on the curve whose y-value is 6.

9 Find the x-value(s) of any point(s) on the curve whose y-value is -4.

10 Find the y-value(s) of any point(s) on the curve whose x-value is 9.

Interviewer Date Student

The 8 points on the graph you see were all graphed by using one algebraic rule. We will call this rule $f(x)$.



1. What is the approximate value of $f(2)$? (About what is y's value when $x = 2$?)
2. For what value of x is $f(x)$ approximately -2.5?
3. What is an approximate solution to the equation $f(x) = 0$?

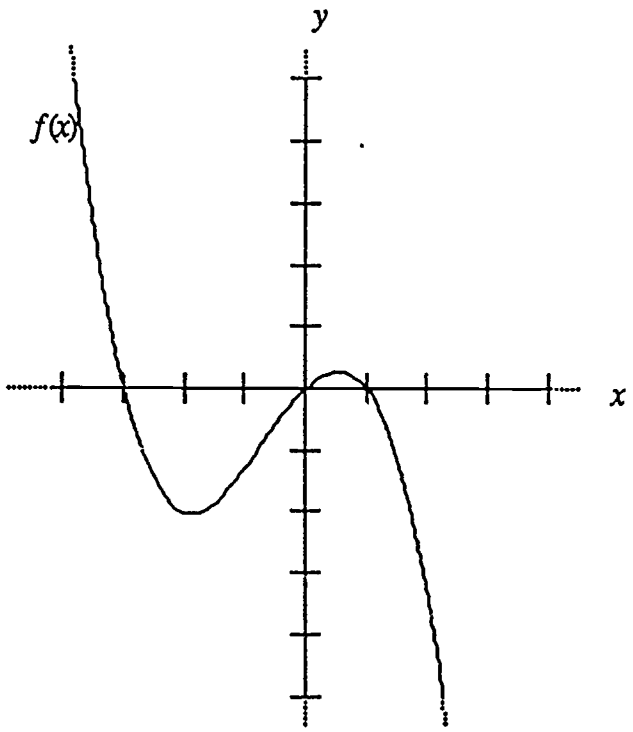
4a. Is it true that the maximum value of $f(x)$ when $x < 0$ is probably 1.

4b. Is it true that when $x < 0$, the maximum value of $f(x)$ is probably greater than 1 and less than 2.

5. Now sketch your best estimate of the complete graph by extending the partial graph above.

Interviewer Date Student

The following graph is a graph of the function $ax^3 + bx^2 + cx + d$, where a, b, c, d are rational numbers. Each mark on the axes is one unit.



1. Is the number a positive, negative, approximately 0 (or is it impossible to tell)?
2. Is the number d positive, negative, approximately 0 (or is it impossible to tell)?
3. How many solutions are there to the equation $ax^3 + bx^2 + cx + d = 0$? (Choose one.)

   none 1 2 3 4 an infinite number

4. How many solutions are there to the equation $ax^3 + bx^2 + cx + d = 5$?

   none 1 2 3 4 an infinite number

5. How many solutions are there to the equation $ax^3 + bx^2 + cx + d = 20$?

   none 1 2 3 4 an infinite number

6. How many solutions are there to the equation $ax^3 + bx^2 + cx + d > 20$?

   none 1 2 3 4 an infinite number

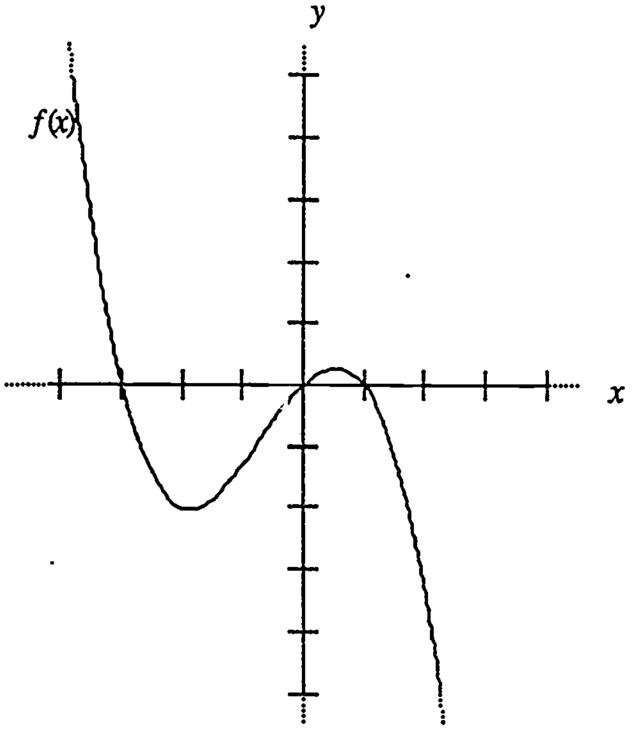
7. Find an approximate solution to the equation $ax^3 + bx^2 + cx + d = -3$.
8. For what number n is there no solution to the equation $ax^3 + bx^2 + cx + d = n$?

Interviewer Date Student

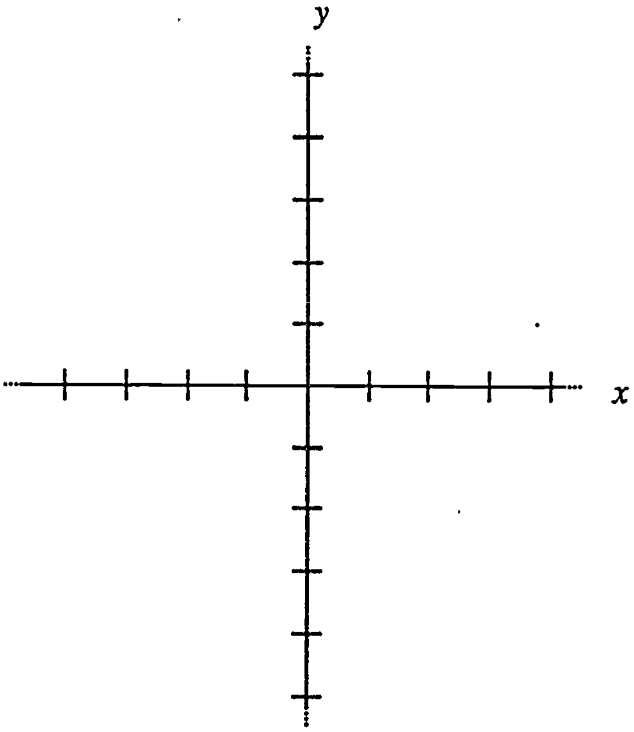
The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.

This set of problems asks you to sketch functions that are closely related to $f(x)$.



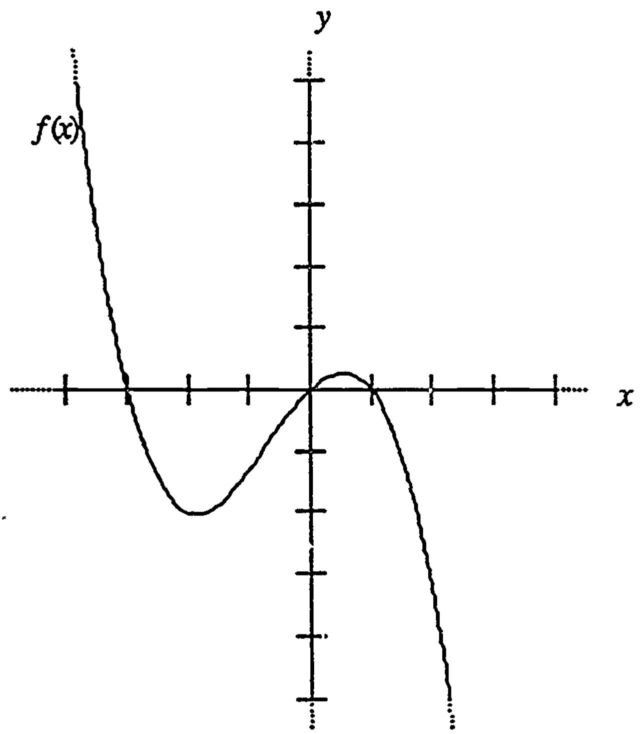
Please sketch the function $f(x)$ - 1 on the blank axes below.



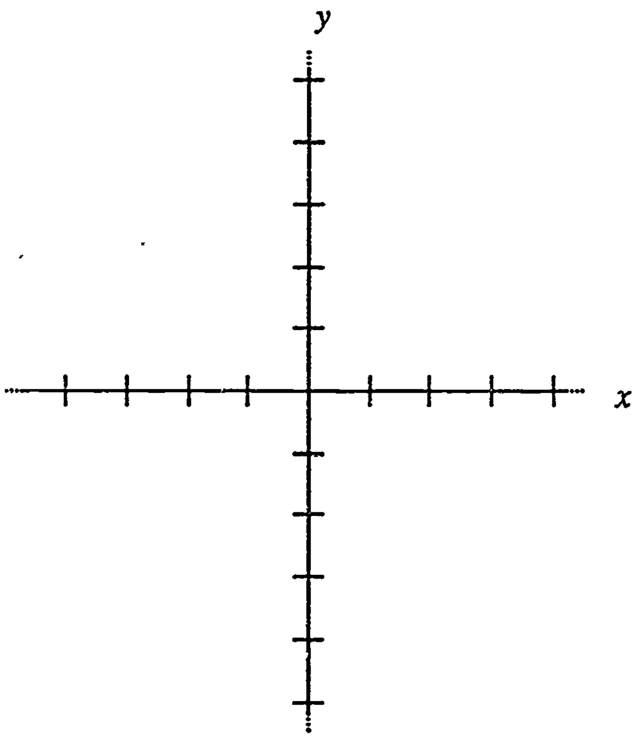
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



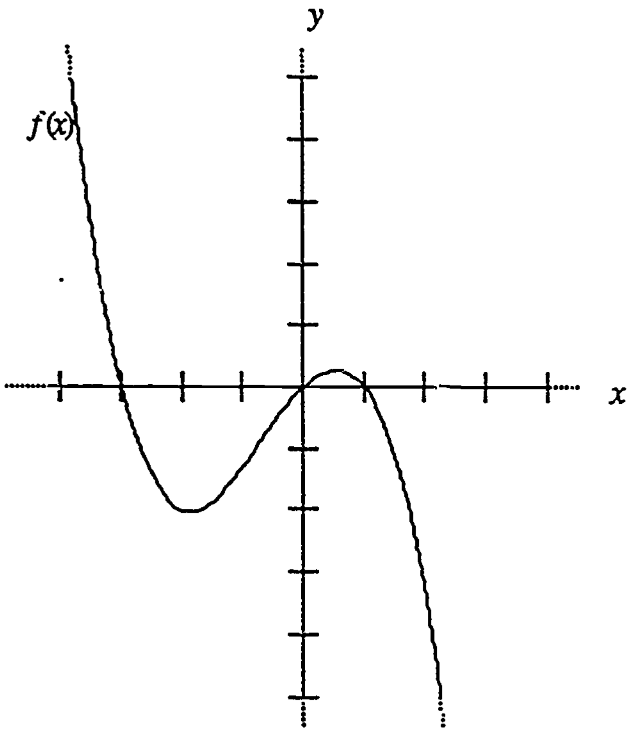
Please sketch the function $f(x - 1)$ on the blank axes below.



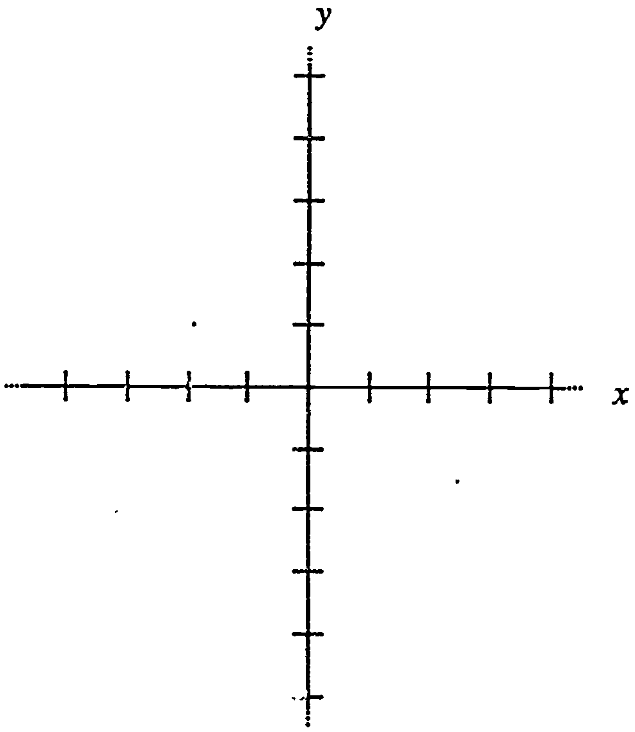
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



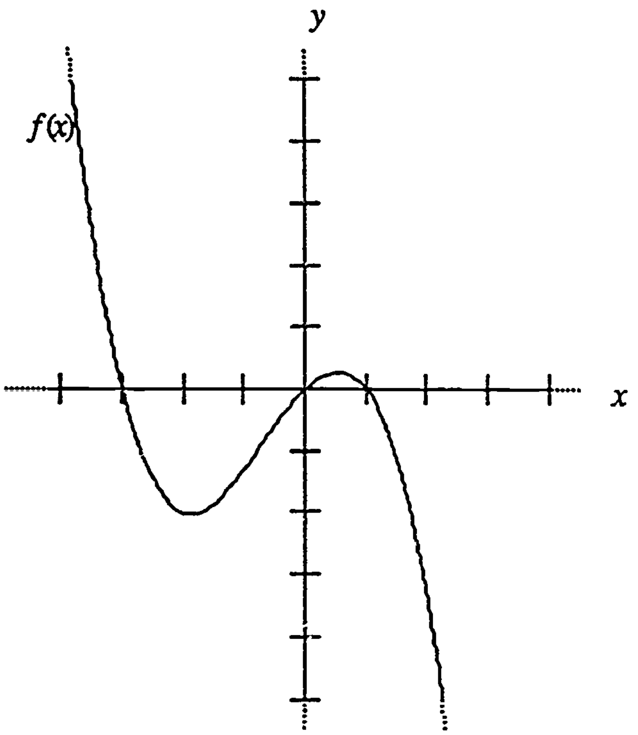
Please sketch the function $f(x) + 1$ on the blank axes below.



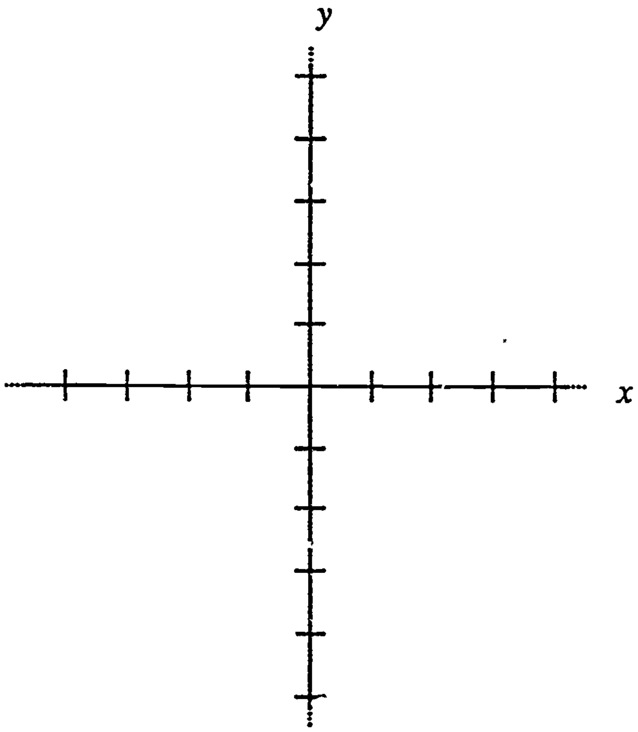
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



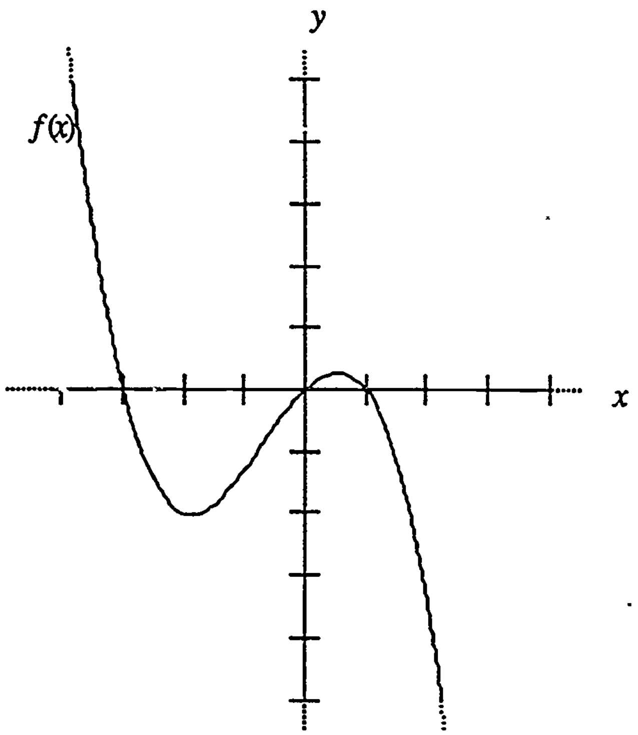
Please sketch the function $f(x + 1)$ on the blank axes below.



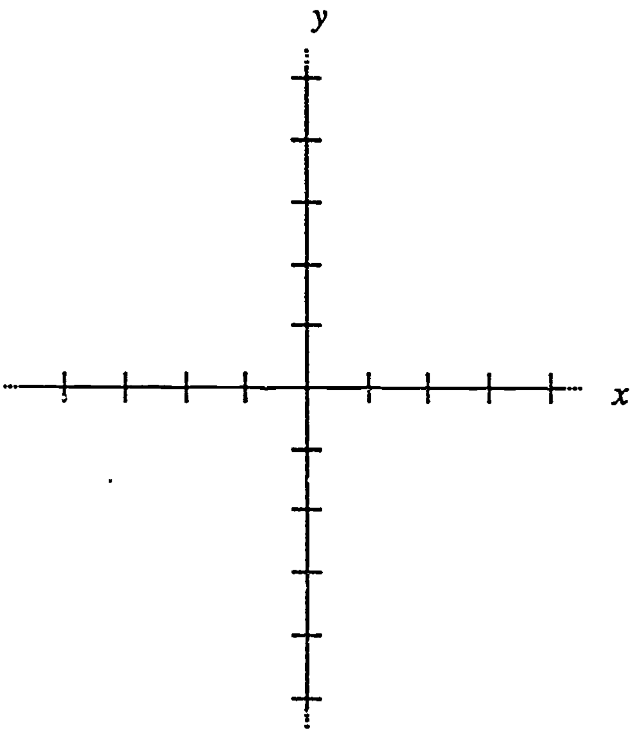
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



Please sketch the function $f(|x|)$ on the blank axes below.



Interviewer

Date

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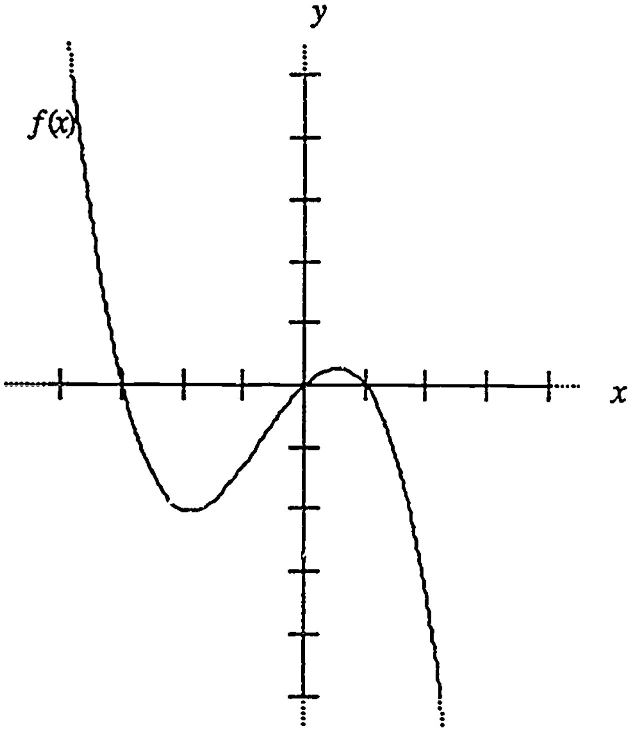
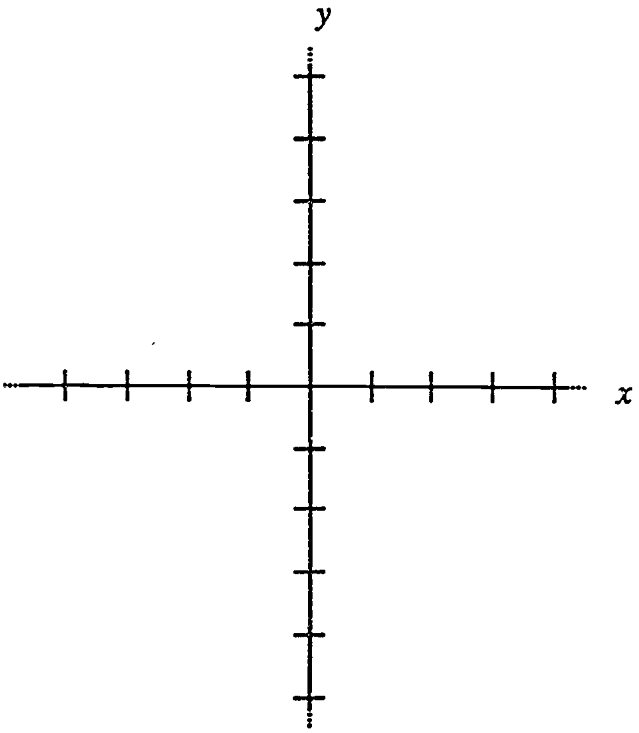
Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.

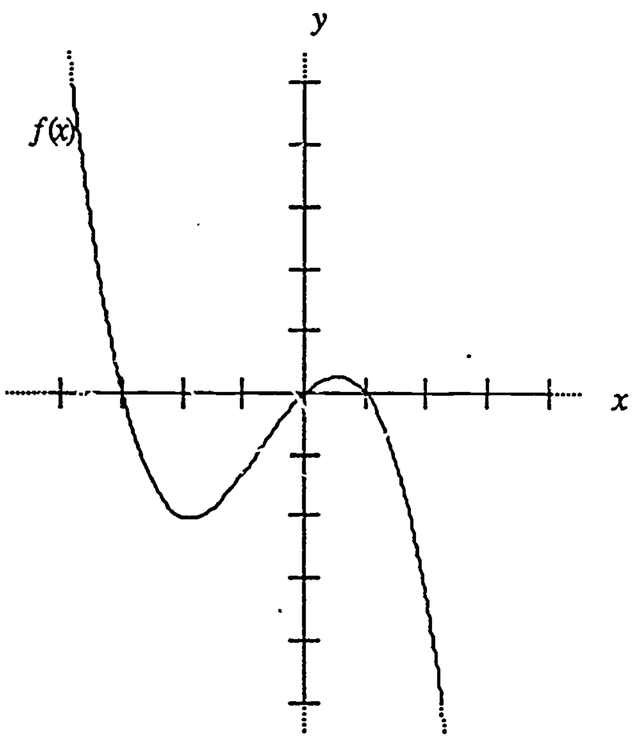


Please sketch the function $|f(x)|$ on the blank axes below.

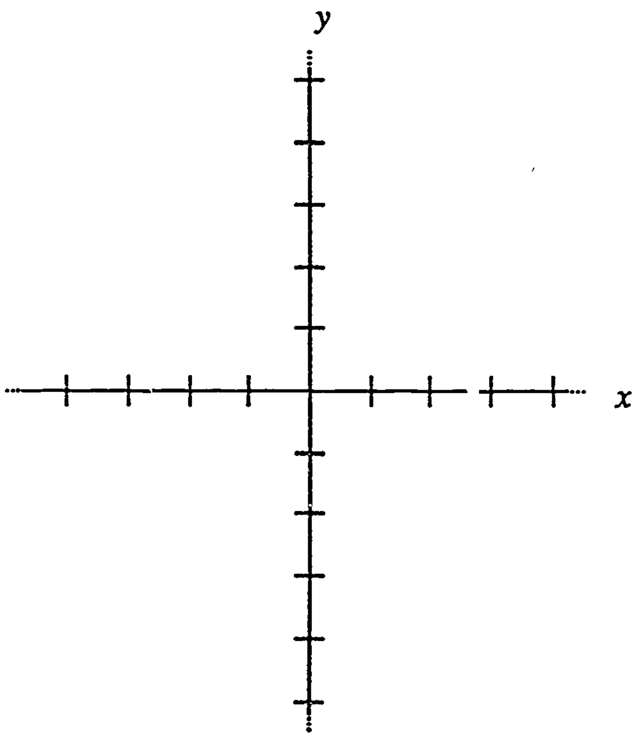


Interviewer ′/4 Date Student

- The graph to the right is a sketch of the function $f(x)$.
- Each mark on the axes is one unit.



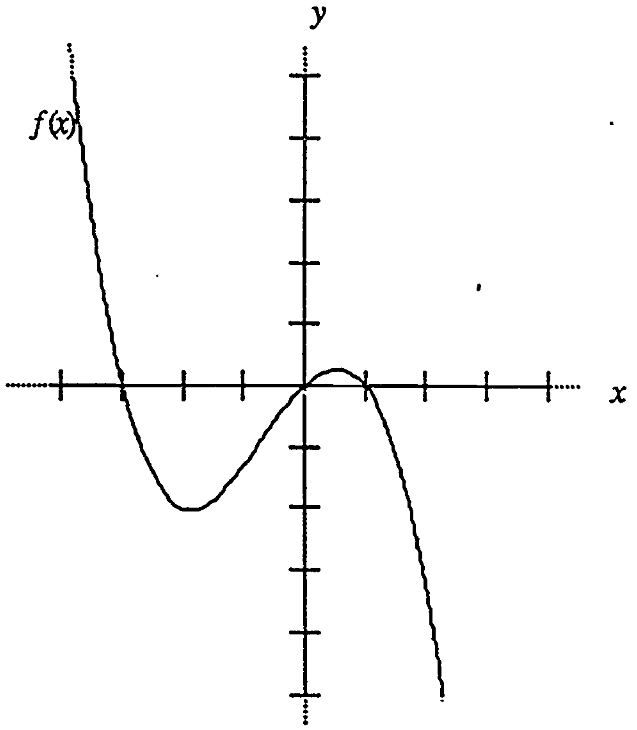
Please sketch the function $-f(x)$ on the blank axes below.



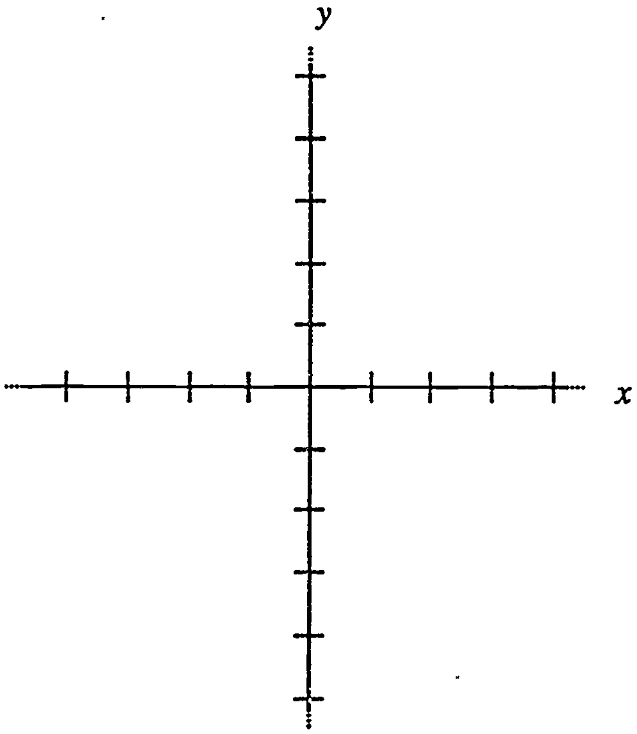
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



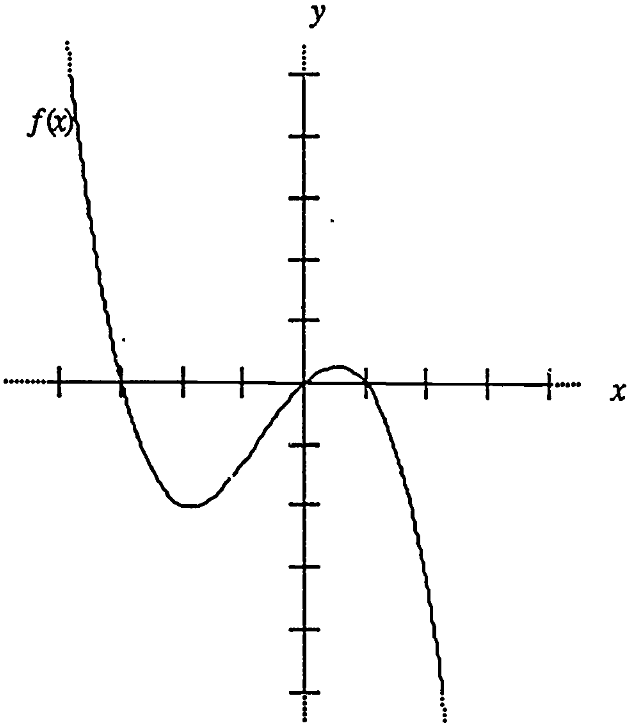
Please sketch the function $f(-x)$ on the blank axes below.



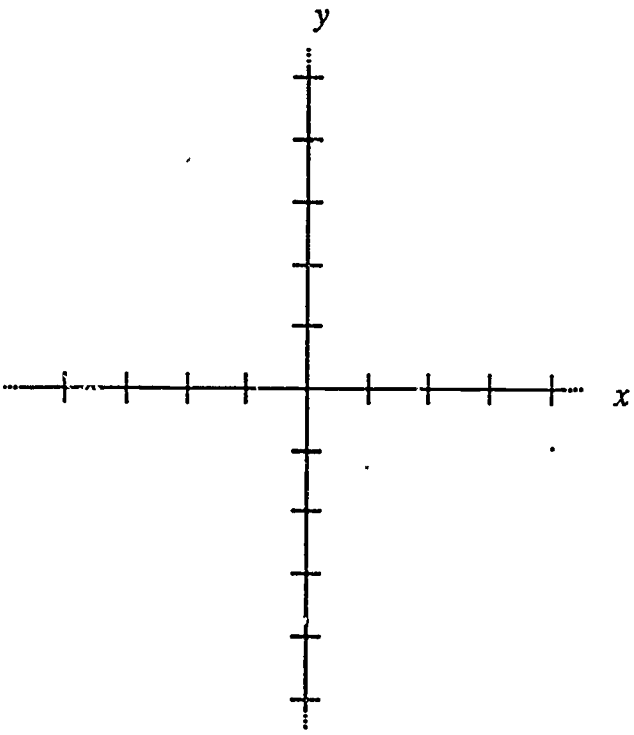
Interviewer 76 Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



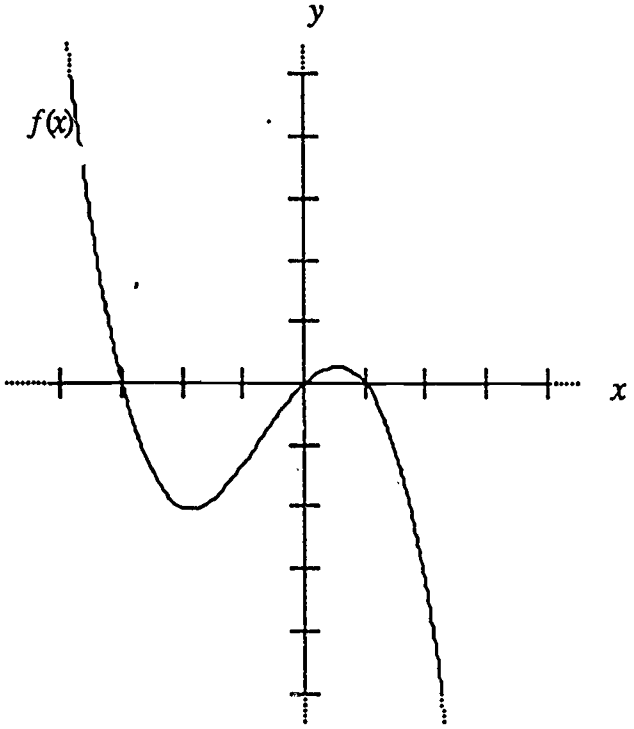
Please sketch the function $f(-|x|)$ on the blank axes below.



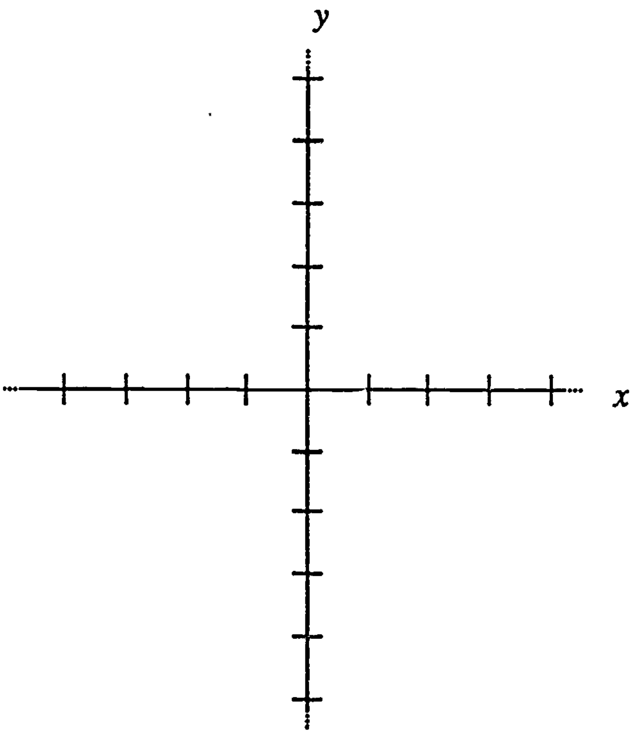
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



Please sketch the function $2f(x)$ on the blank axes below.



Interviewer Date Student

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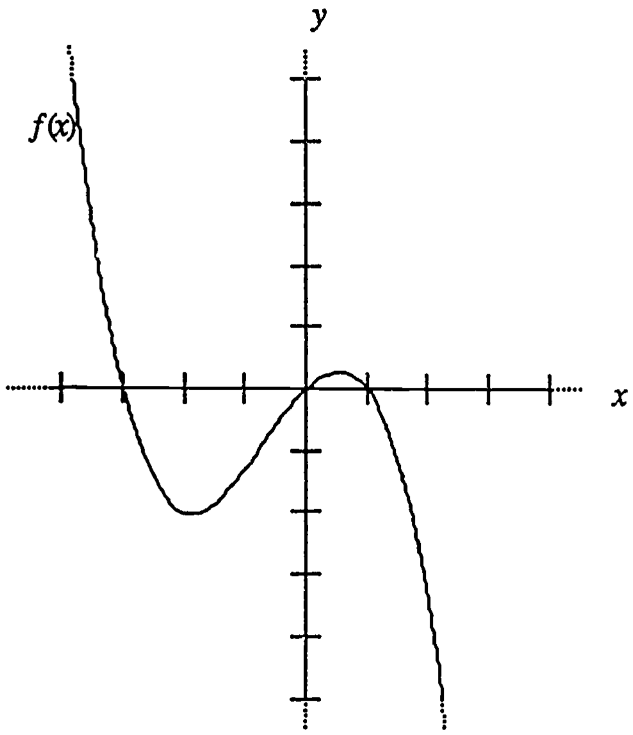
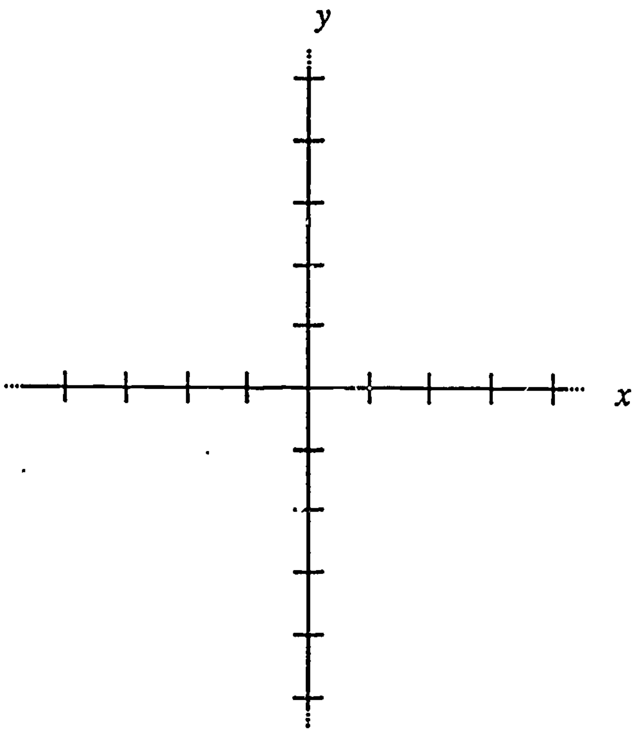
- The graph to the right is a sketch of the function $f(x)$.
- Each mark on the axes is one unit.



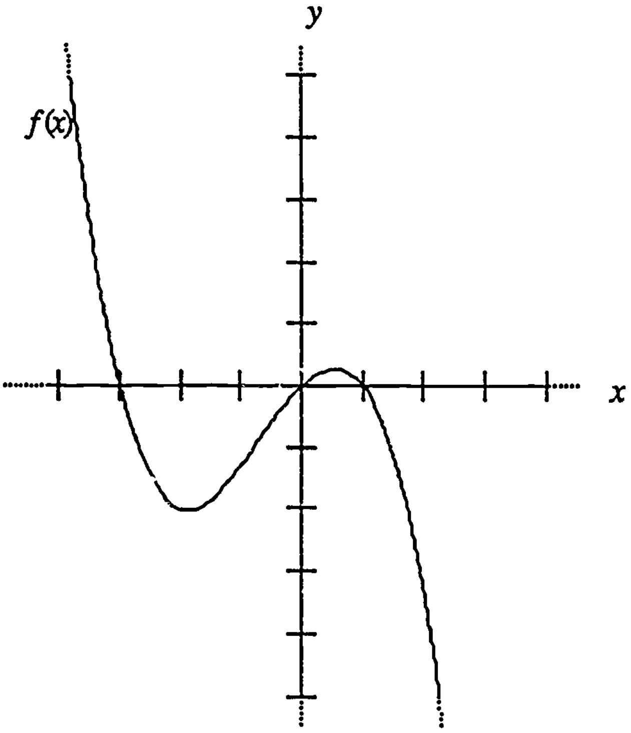
Please sketch the function $f(2x)$ on the blank axes below.



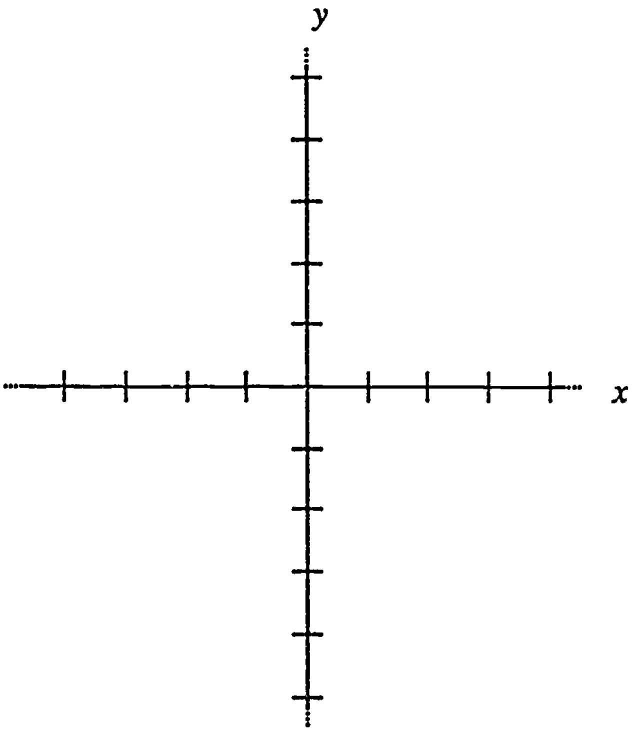
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



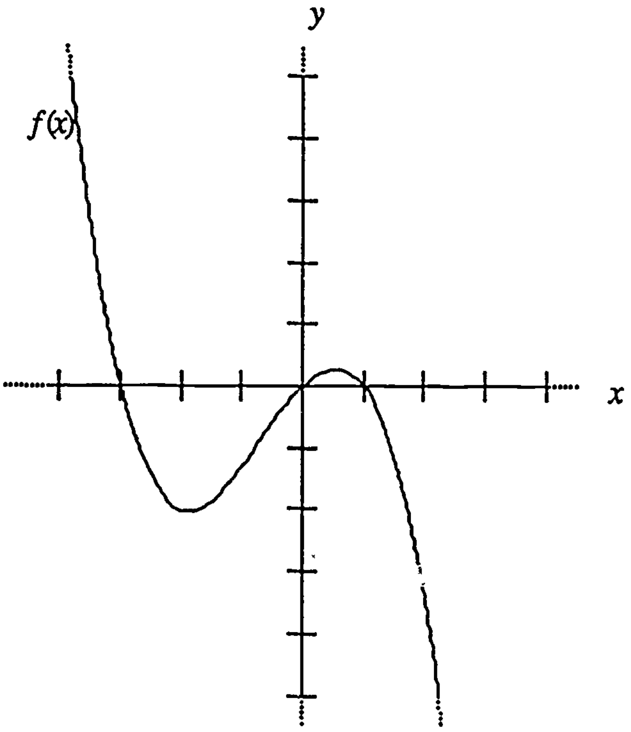
Please sketch the function $\frac{1}{2}f(x)$ on the blank axes below.



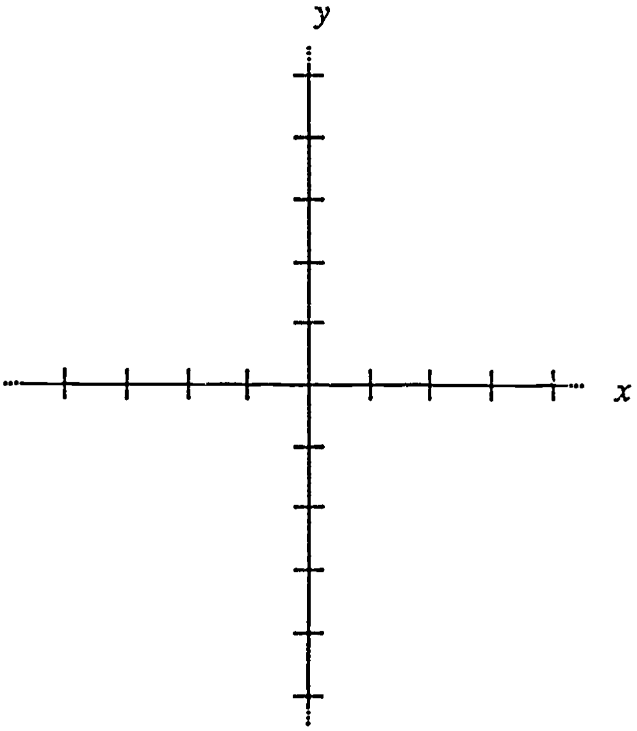
Interviewer Date Student

The graph to the right is a sketch of the function $f(x)$.

Each mark on the axes is one unit.



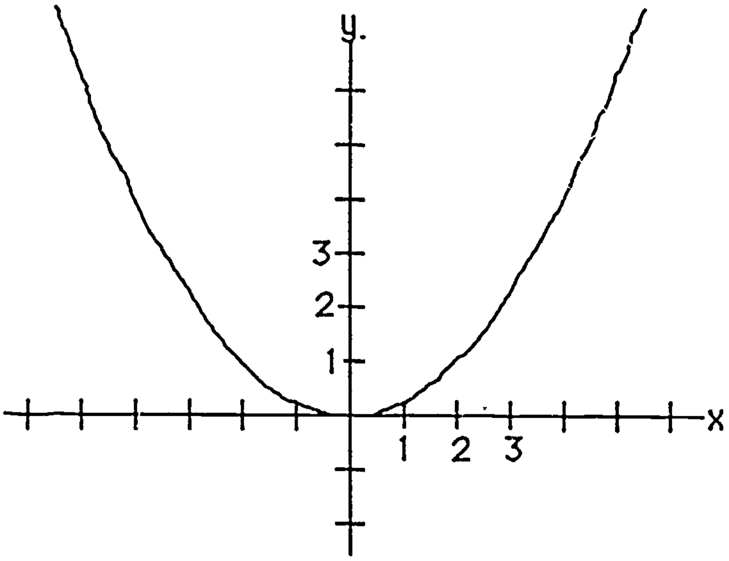
Please sketch the function $f(\frac{1}{2}x)$ on the blank axes below.



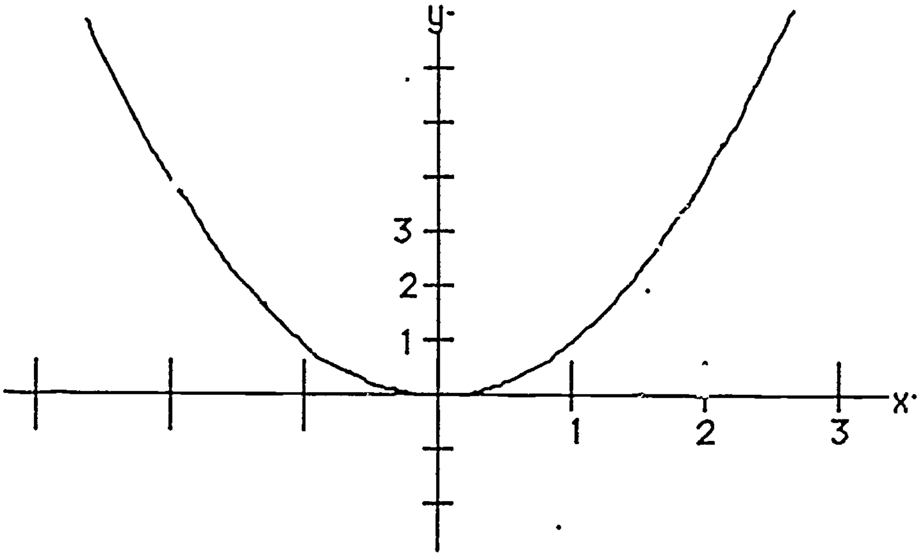
Interviewer Date Student

The two graphs below are both parabolas.

Parabola 1:



Parabola 2:



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Interviewer Date Student

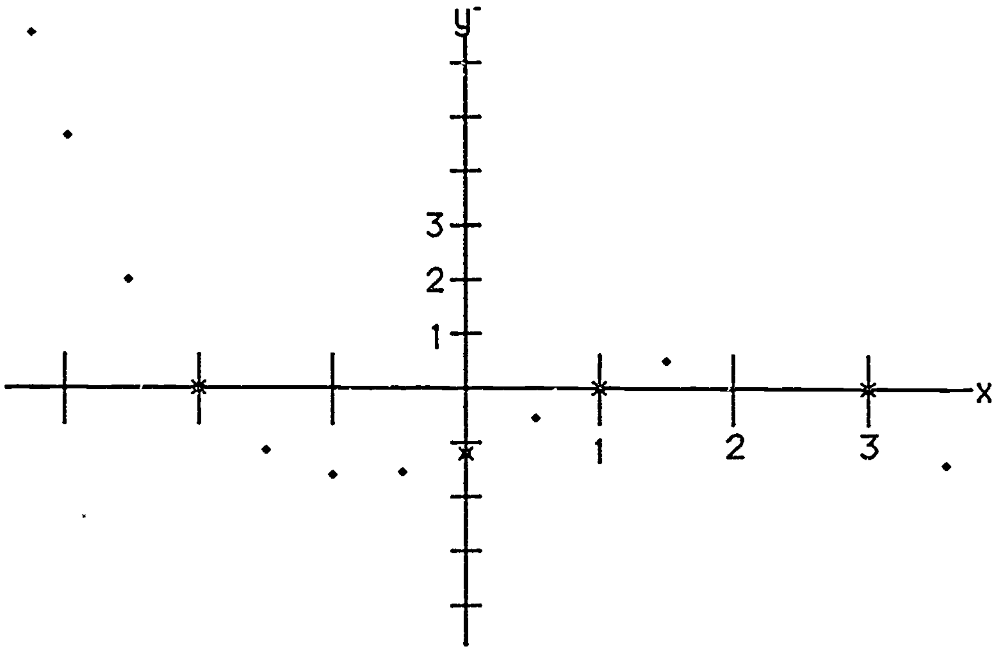
Establish a language with the student.

If function notation can be used, label parabola 1 $f(x)$ and parabola 2 $g(x)$. Otherwise, label them y_1 and y_2. Change the language of the questions below accordingly.

1. Which parabola rises more steeply? Explain your reasoning.	1. Which parabola rises more steeply? Explain your reasoning.
2. Which is smaller, $f(\frac{1}{2})$ or $g(\frac{1}{2})$?	2. Which is smaller when $x = \frac{1}{2}$, y_1 or y_2?
3. Could $f(x)$ and $g(x)$ be described by the same algebraic rule? Explain your answer.	3. Could the two parabolas be graphs of the same algebraic rule? Explain your answer.
4. Plot the graph of $f(x)$ on the same axis as that of $g(x)$.	4. Plot the graph of parabola 1 on the same axis as that of parabola 2.
5. Now plot the graph of $g(x)$ on the same axis as that of $f(x)$.	5. Now do the reverse. Plot the graph of parabola 2 on the same axis as that of parabola 1.

Interviewer Date Student

The partial graph of the function $f(x)$ below contains 13 points.



1. Make a table of approximate values corresponding to these points:

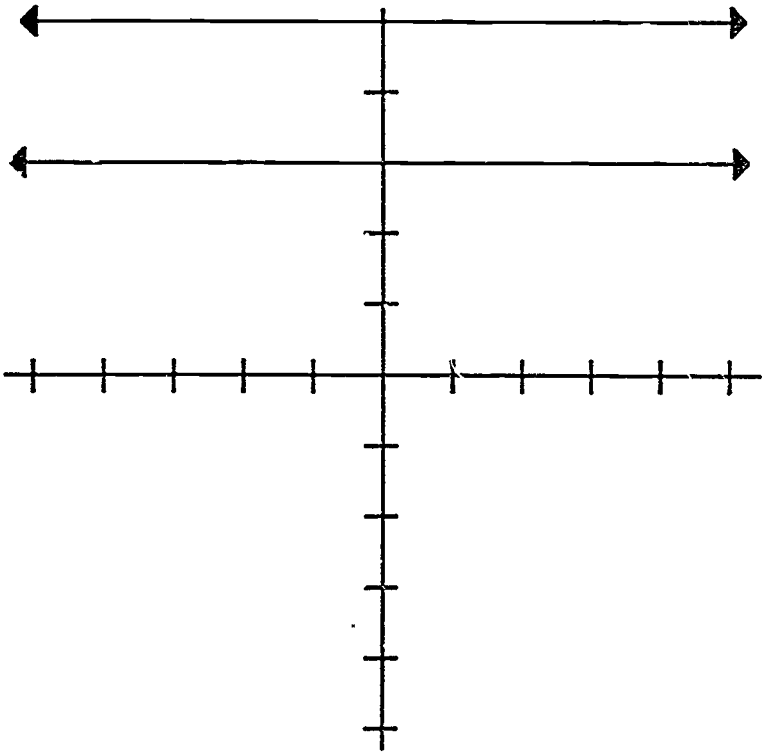
x	y

2. Between which adjacent pair of these points is the value of the function changing the least?

3. Between which adjacent pair of these points is the value of the function changing the most?

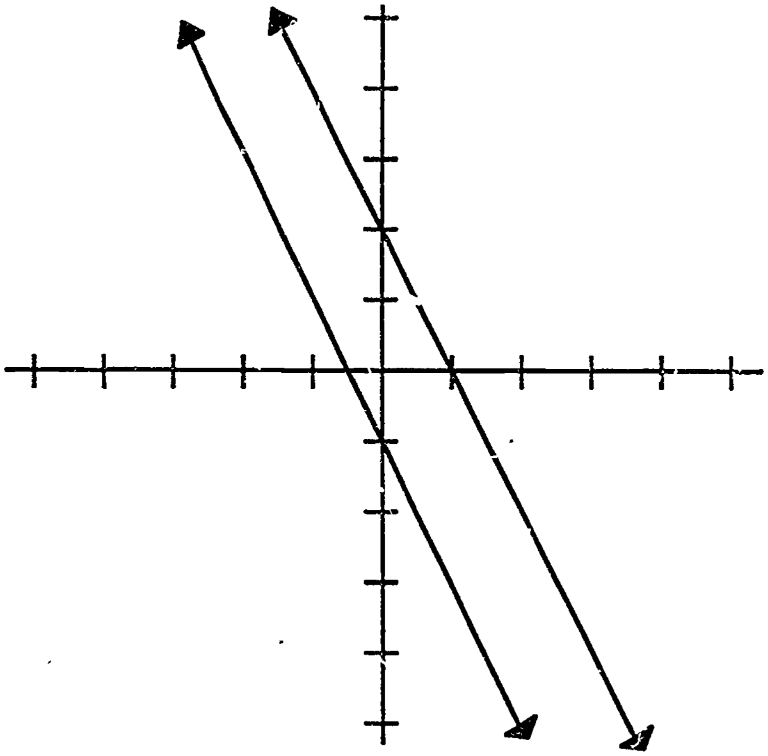
Interviewer Date Student

1. Below are the graphs of $f(x) = 10$ and $g(x) = 6$. On the same set of axes, sketch a graph of their difference, $h(x) = f(x) - g(x) = 4$.



Interviewer Date Student

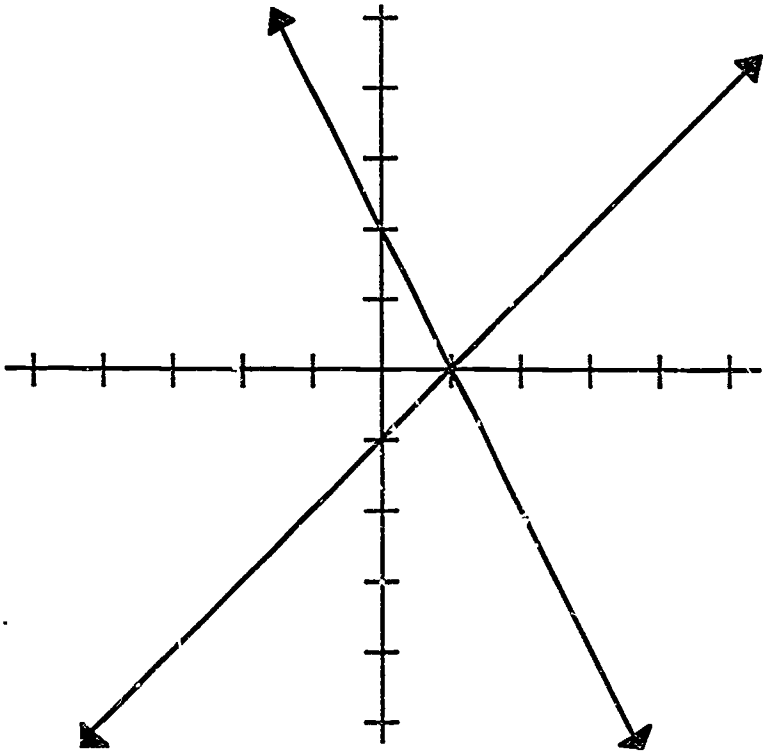
2. Below are the graphs of $f(x) = -2x - 3$ and $g(x) = -2x + 6$. On the same set of axes, sketch the graph of their difference, $h(x) = f(x) - g(x) = -9$.



4. Which is greater, $f(x)$ or $g(x)$? Explain how you know.

Interviewer Date Student

5. Below are the graphs of $f(x) = -2x + 6$ and $g(x) = x - 3$. On the same set of axes, sketch the graph of their sum, $h(x) = f(x) + g(x) = -x + 3$.



6. For what values of x is $f(x)$ greater than $g(x)$? Explain how you arrive at your conclusion.

Interviewer Date Student

1. Find all x which satisfy: $3x - 5 = 2x - 4$. Explain your method.

2. Find all x which satisfy: $2x - 5 = 2x - 4$. Explain your method.

3. Find all x which satisfy: $x^2 + 5x + 6 = (x + 2)(x + 3)$. Explain your method.

4. Evaluate:

a) $10 - 6$

b) $3 - (-2)$

c) $-2 - 5$

5. Find the difference between:

a) 12 and 19

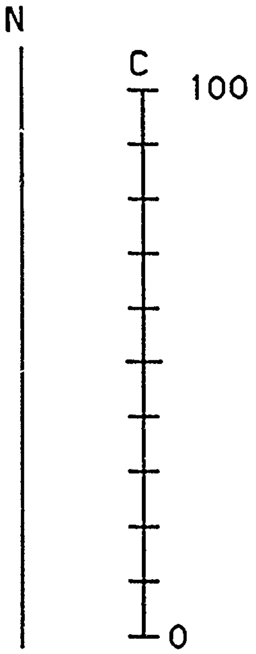
b) 6 and -7

c) -15 and -9

Interviewer Date Student

APPENDIX C

SCALE



The scale at the far right represents a diagram of the Celsius thermometer scale. 100° C is the boiling point of water.

0° C is the freezing point of water.

On the line to the left of the Celsius scale you will be asked to sketch a similar diagram of the Nory scale.

The Nories are an 8 fingered race dwelling on a distant planet. On the Nory thermometer, 0° N is the freezing point of water and 8° N is the boiling point of water. Show how the Nory scale might be marked in Nory degrees.

1 What is 0° N in Celsius measure? Draw an arrow from the point on the Nory scale to the matching point on the C scale.

2 What is 100° C in Nory measure. Draw an arrow from the C point to the matching N point.

3 What is 4° N in C measure? ________

4 What is 25° C in N measure? ________

5 What is 6° N in C measure? ________

6 What is 16° N in C measure? ________

7 What is 110° C in N measure? ________

8 What is 9° N in C measure? ________

9 Which of the following formulas provides a correct conversion from Nory to Celsius measures?

a) $^\circ C = 25/2\,(^\circ N)$

b) $^\circ N = 2 \bullet {}^\circ C / 25$

c) $^\circ C = 50 \bullet {}^\circ N$

d) $^\circ N = {}^\circ C / 12$

10 What is 2° N in Celsius measure? ____________

11 What is 125° C in Nory measure? ____________

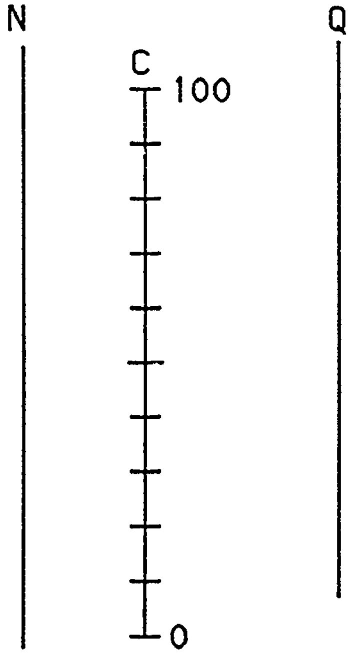
Interviewer Date Student

Reproduce your N scale from the previous sheet, Scale 1a.

You will be asked to draw another scale, the Qaos scale on the line at the right.

Lord Qaos was a Nory. He agreed that the spacing of the Nory scale was correct because the 8 spaces between the boiling point of water and the freezing point of water were what The Maker of All intended. After all, not only water but also Plenbium, the most precious of the eight liquid metals, boiled at exactly 8 Nory degrees higher temperature than its freezing point. It freezes at 3° N and boils at 11° N.

Lord Qaos decided that Plenbium and not water should be honored by having its freezing point called 0. Accordingly he created the Q scale which maintained the N-scale divisions, but set 0° Q to be the freezing point of Plenbium.



1 Sketch the Q scale including the divisions.

2 What is 100 N on the Q scale? __________

3 What is 0 C on the Q scale? __________

4 What is 8 Q on the C scale? __________

5 What is 4 Q on the N scale? __________

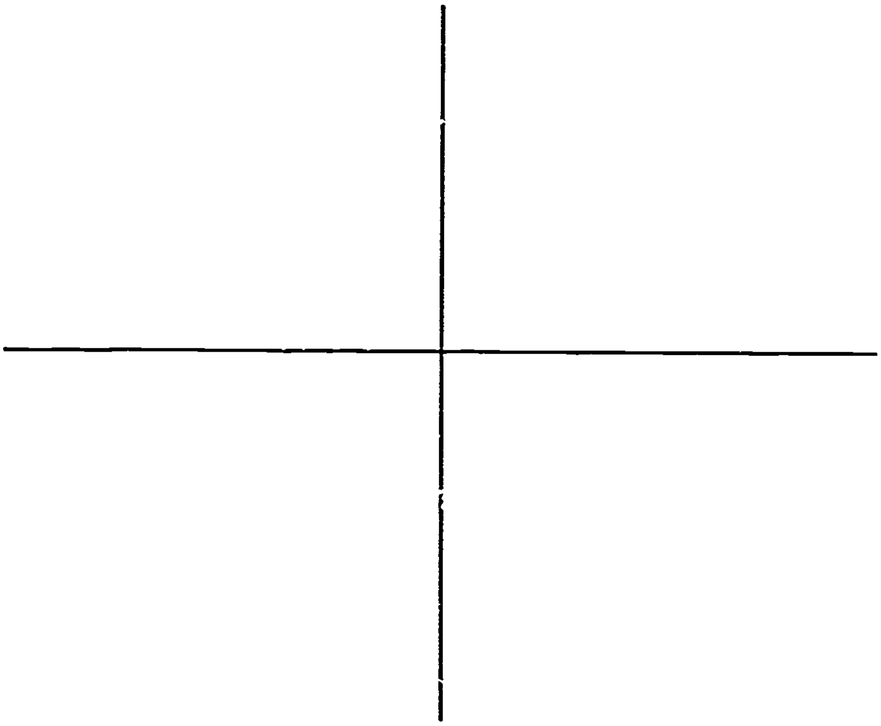
6 What is 1 Q on the C scale? __________

7 What is 1C on the Q scale? __________

Interviewer Date Student

(To be conducted *only* in interview form. The wording here was chosen to suggest the purpose and tone, and is not a prescribed wording.)

You have probably done some graphing on this kind of graph paper. Am I right?



Would you point to the horizontal axis just so I know we are looking at the same thing?

Make some marks on the horizontal axis.

Would it make sense to you to put numbers on the horizontal axis so that it would represent the Nory scale? Go ahead and do that and label it with an N.

Make some marks on the vertical axis.

Could this axis repesent the Celsius scale? Put numbers on it and label it with a C.

Interviewer Date Student

Collect two or three matching pairs from your diagram on page 1 and enter them in this table:

N	C

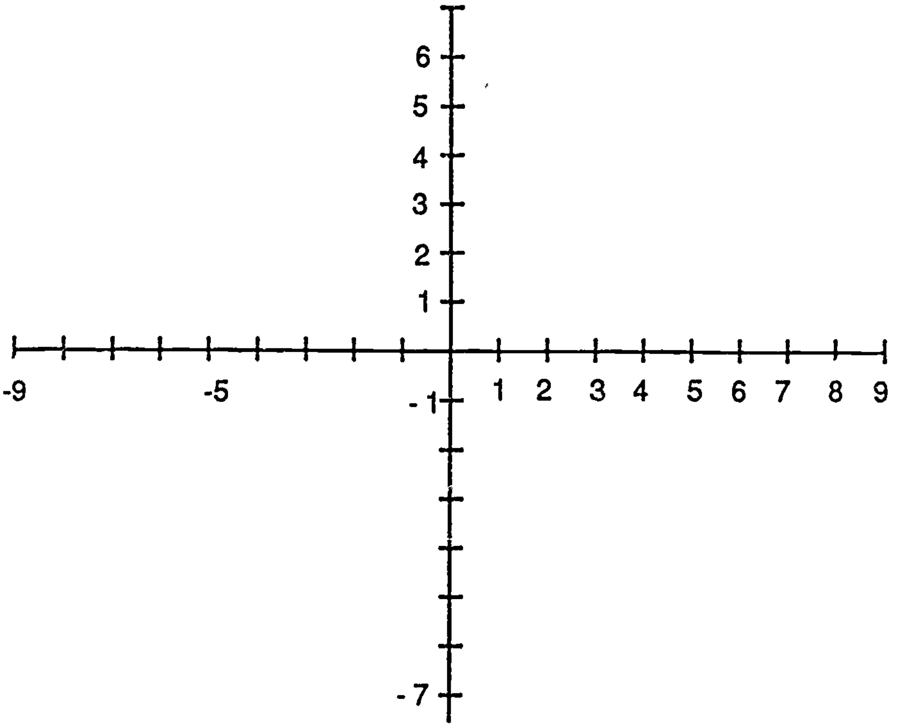
Record the information indicated in your table on the axes above. (e.g., How would you indicate that 4° Nory is the same as 50 degrees Celsius?)

Does this graph help determine the Celsius measure for 3° N? (How?)

What is 70° C in Nory measure? (How does the student answer this question? Using the graph? Formula? Refering to earlier probe?)

Interviewer Date Student

Could you have used this grid to record some of the information relating Nories to Celsius?



Describe the differences in the results of the two graphs.

Interviewer Date Student

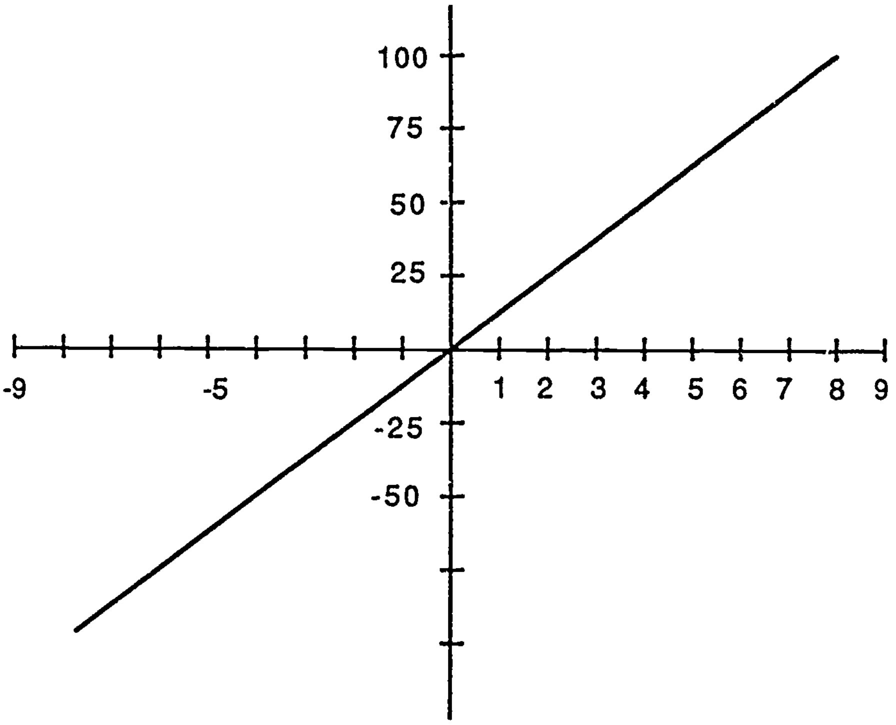
Below is a graph of the temperature in degrees Celsius vs. the temperature in degrees on the Nory scale. Does it look right to you?

Which axis represents the number of degrees Celsius?

Can you produce a graph on these same axes that indicates the Celsius temperature vs. the Qaos temperature?

If you can, please do it.

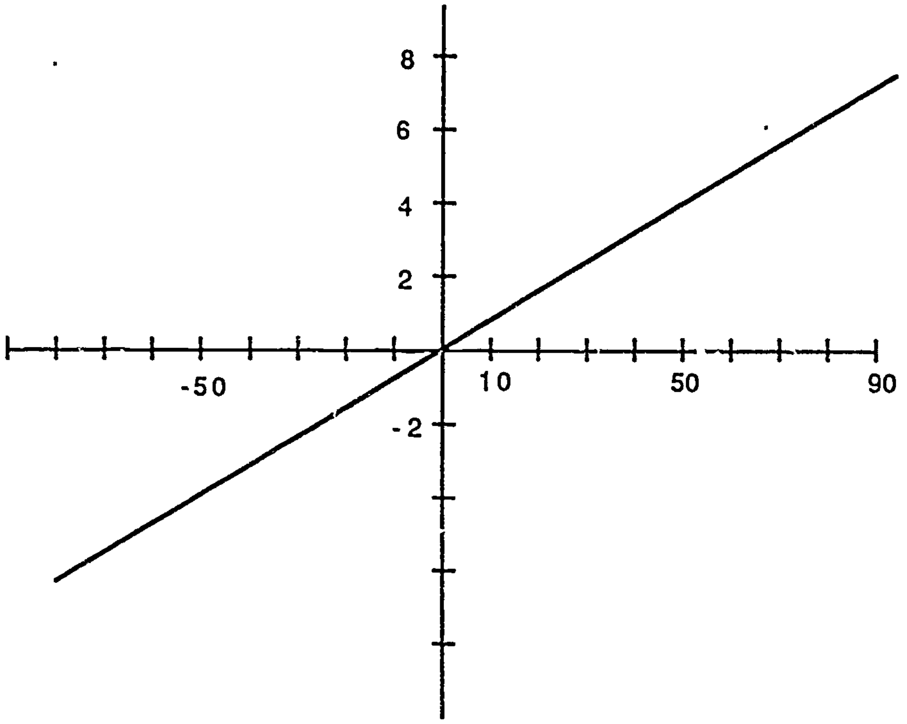
Would it help to label the axes? [C and N (or Q?) as before]



Interviewer Date Student

Someone made this graph to compare the temperature in Celsius with that in Nories.

Did they do it correctly?

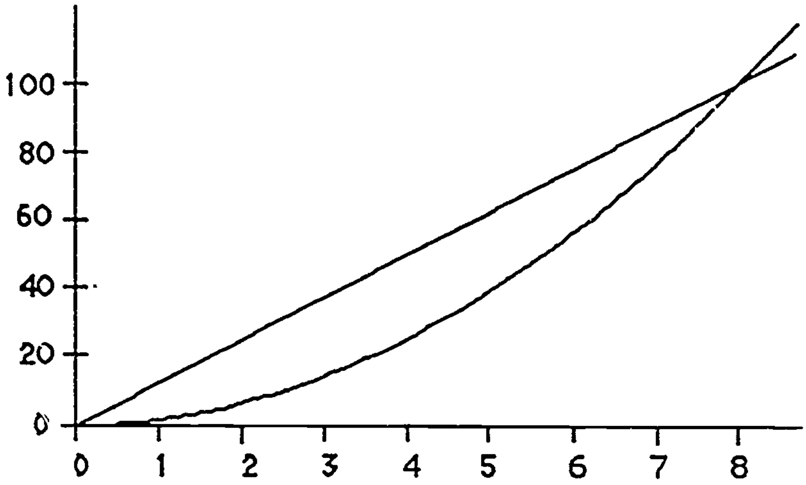


96

Interviewer Date Student

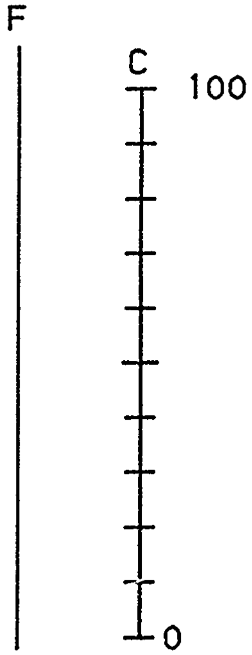
The straight line here is another graph of Celsius temperature vs. Nory temperature. Does it look correct?

The curve represents Celsius plotted against yet another scale, the Flauntium Scale.



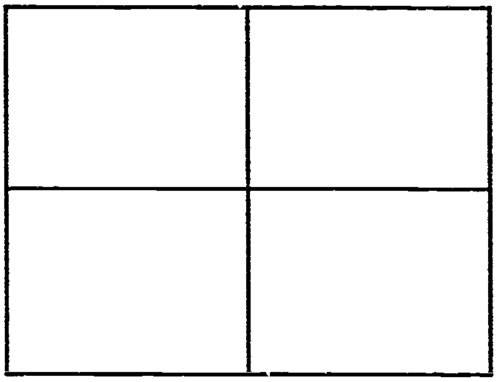
F	C

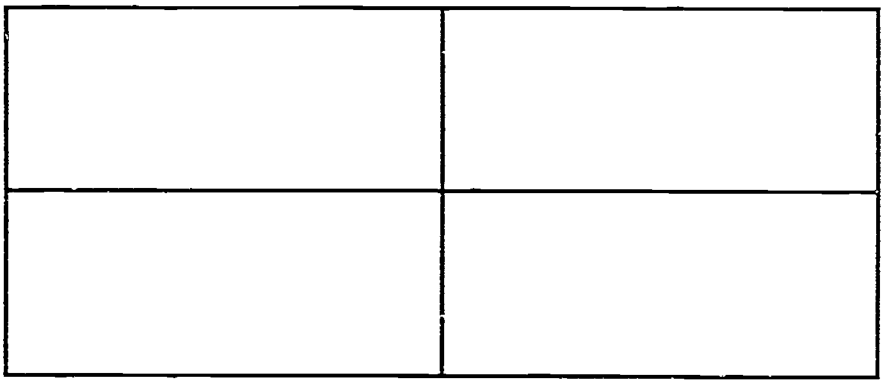
Take a few data points from the graph above and enter them in the table at the right.



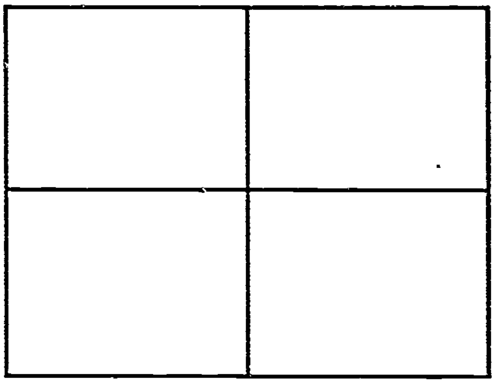
Draw the scales in this case, with the Flauntium and Celsius scales next to each other.

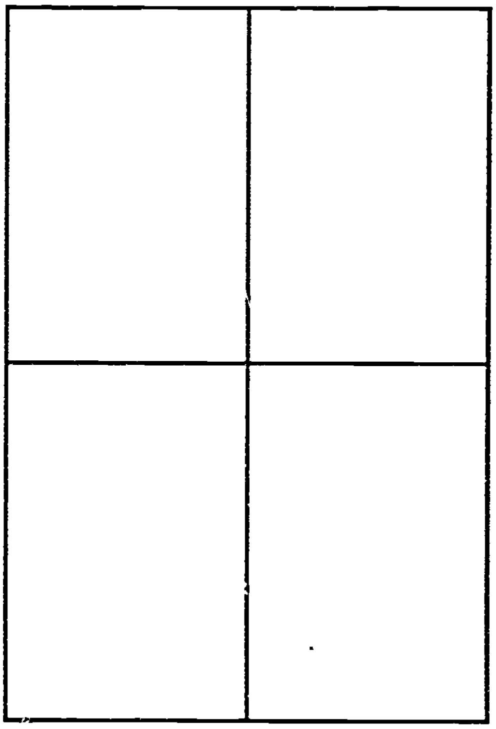
Interviewer Date Student





Interviewer Date Student





Interviewer Date Student

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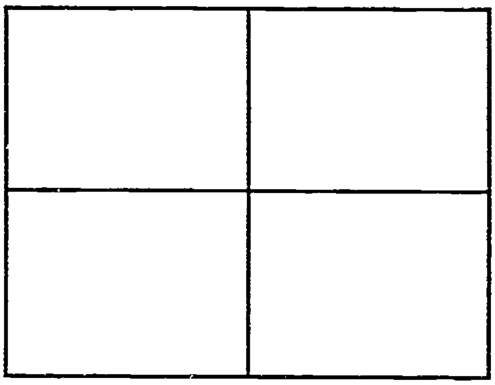
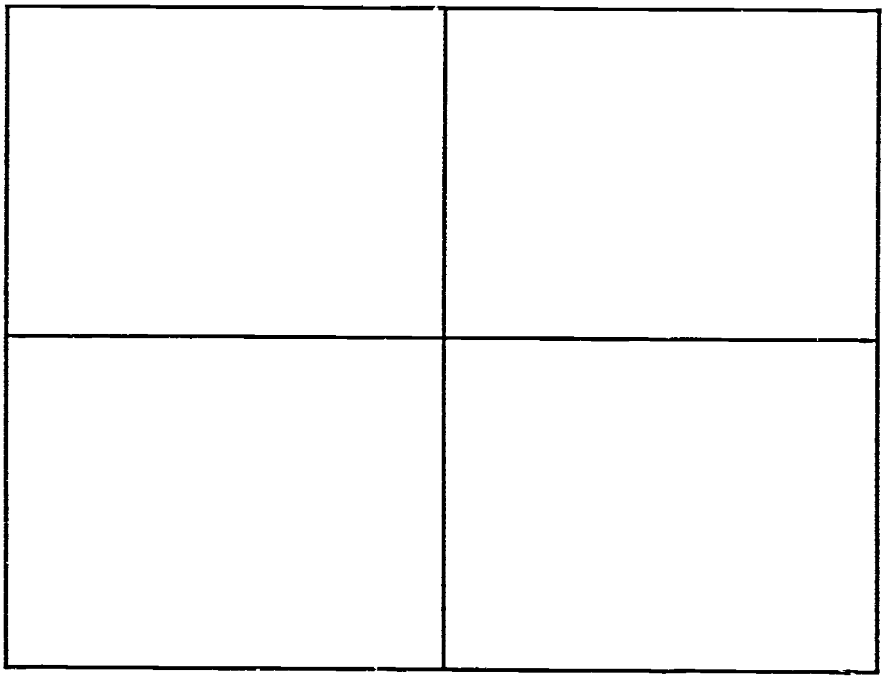
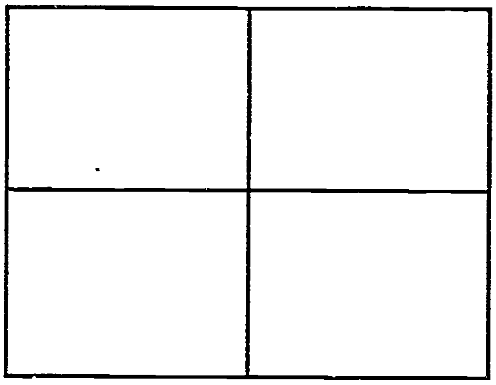
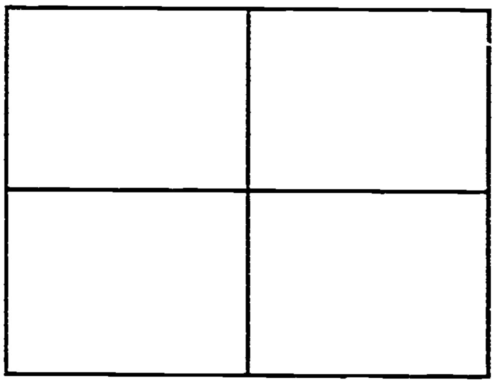




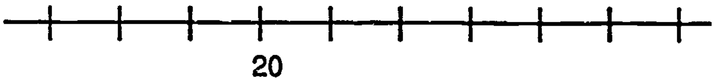
Interviewer Date Student



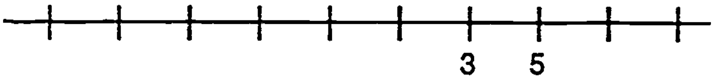


Interviewer Date Student

1 On the axis below one point is marked. Locate and mark positions for 30, 25, and 0.

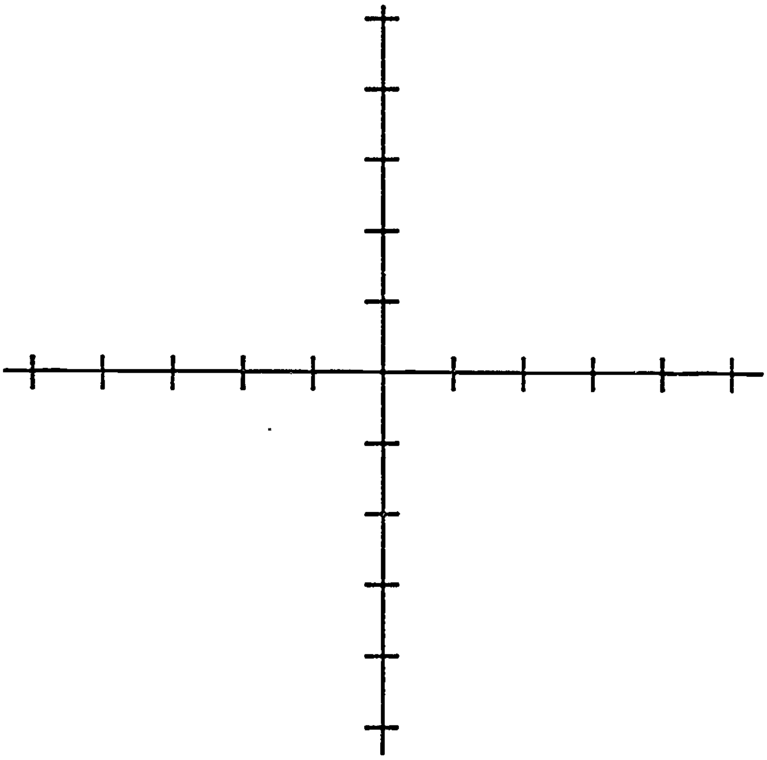


2 On this axis two points are marked. Locate and mark the positions for 6, 9, 0, and -3.



3 Use the axes provided to help plot the following points. You should choose a scale that will allow you to graph all of the given points. Label points on the axes with numbers to indicate the scale that you have selected.

A = (40,50) B = (-25, 20) C = (7, 0) D = (-2, -12)



Interviewer Date Student

Examine this graph.

Is it possible to mark the x and y axes so that the graph can represent this function?

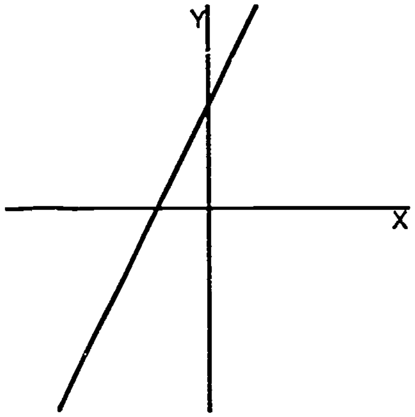
In successive questions, the interviewer shows each function below one by one.

a $y = 3x + 4$

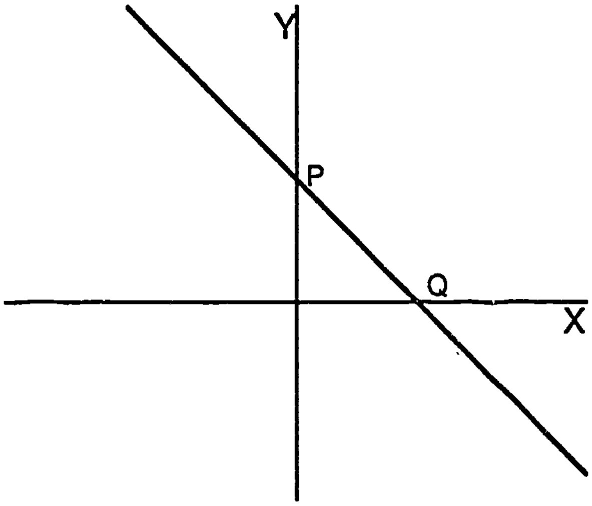
b $y = 2x + 6$

c $y = x - 3$

d $y = -3x + 2$

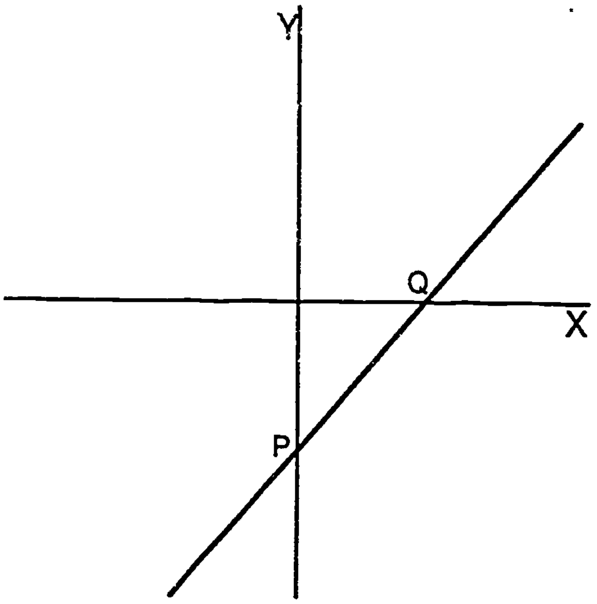


The graph at the right shows the function $y = -x + 8$. Determine the coordinates of the points labeled P and Q.



Interviewer Date Student

Here is a graph of $y = 2x - 8$. Determine the coordinates of the points labeled *P* and *Q*.



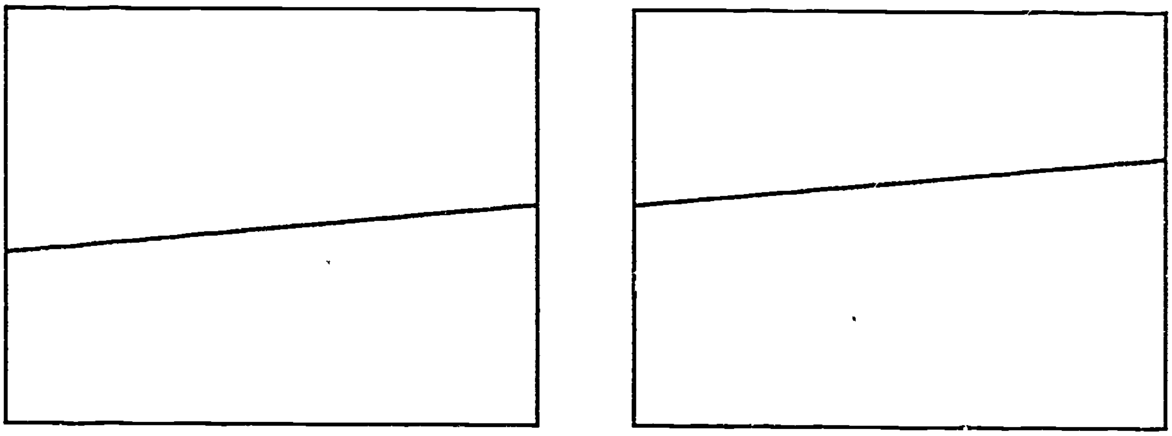
Interviewer Date Student

Normal conventions of graphing in the Cartesian coordinate plane allow for a restricted group of scale changes.

1. The two axes must remain perpendicular to each other, one designated to be horizontal the other vertical.
2. Lower numbers are placed at the left side of the horizontal axis and the bottom of the vertical.
3. Units may be chosen independently for the two axes. That is, knowing that a given length of the horizontal axis represents 20 units gives us no information about the number of units represented by the same length of vertical axis.
4. On a particular axis the unit value is fixed. That is, it can be assumed that if a given length of axis represents 30 units, then each thirtieth of that length is equivalent to one unit, each 300th of the length is equivalent to one tenth unit, etc..

Assuming the above restrictions on scale changes, which of the following pairs of graphs could be differently scaled views of the same line?

1.



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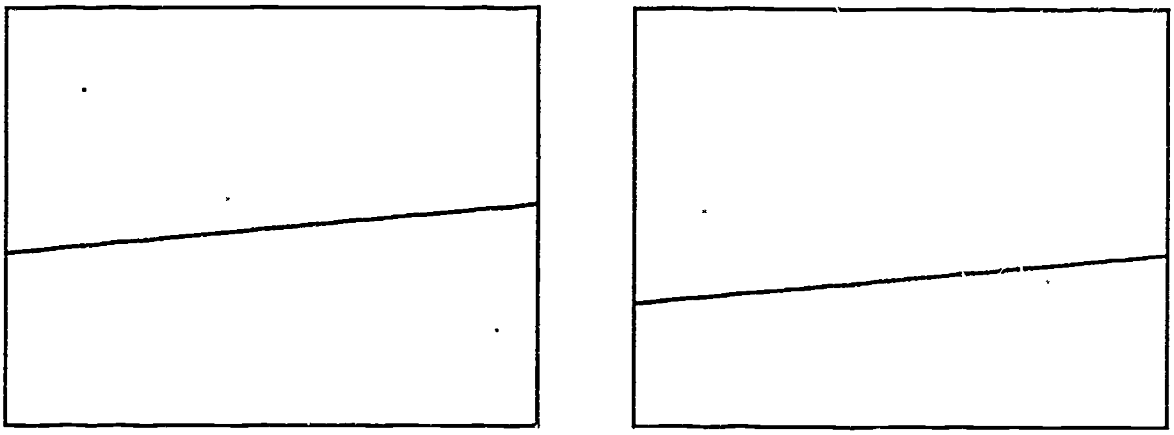

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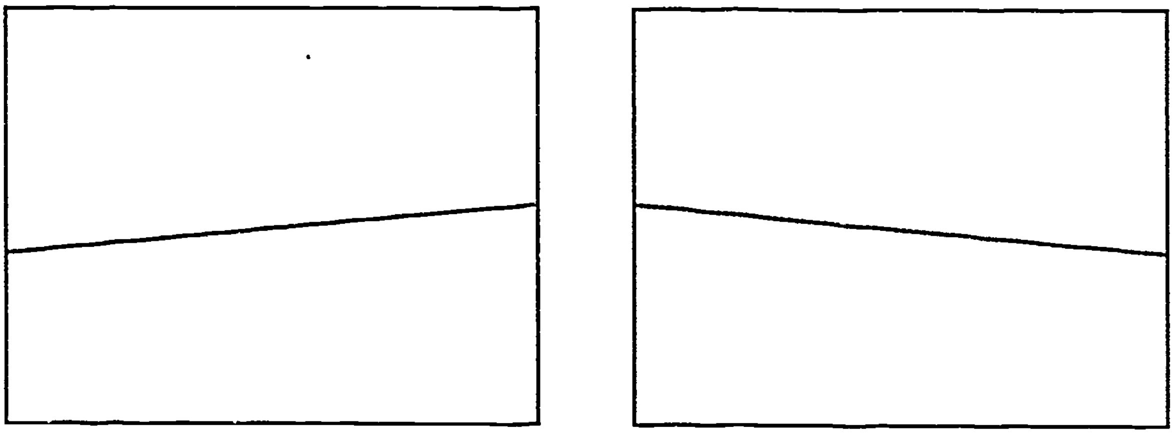

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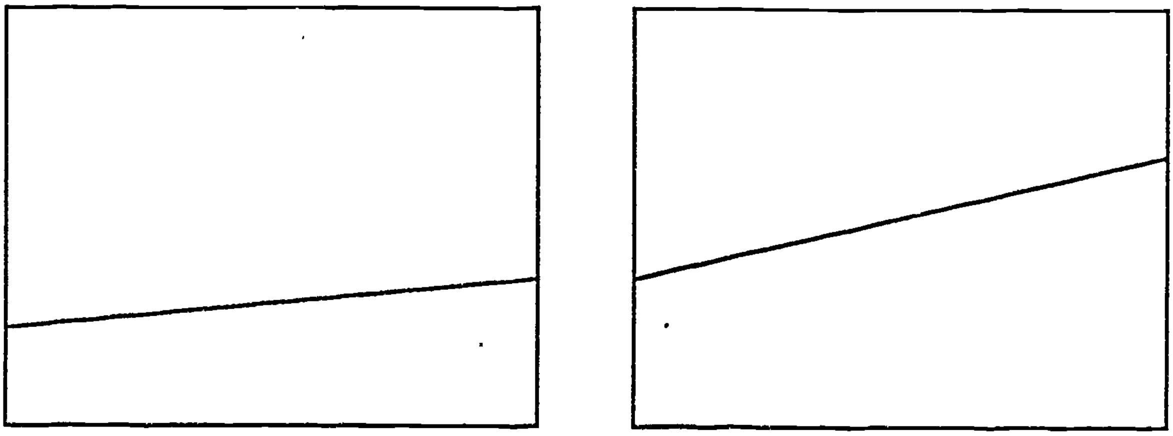
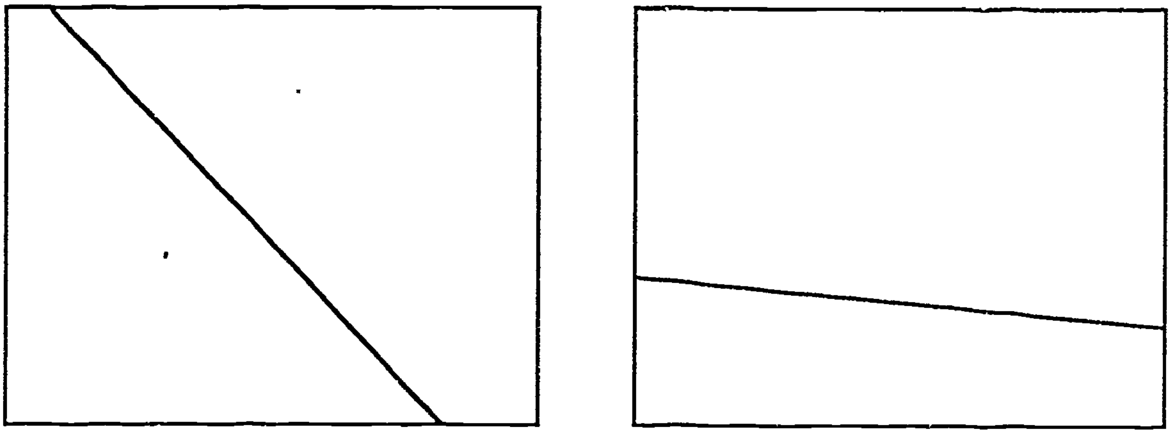


Interviewer Date Student

5.



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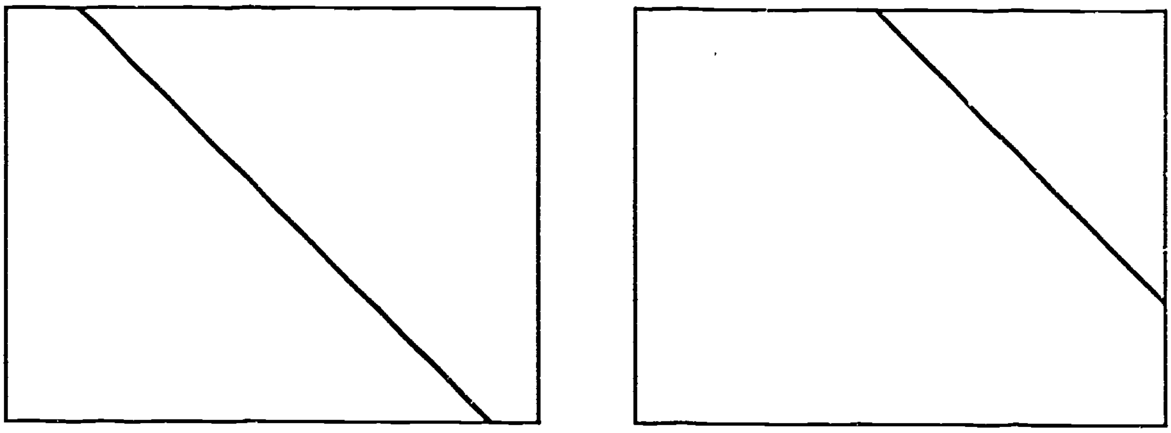

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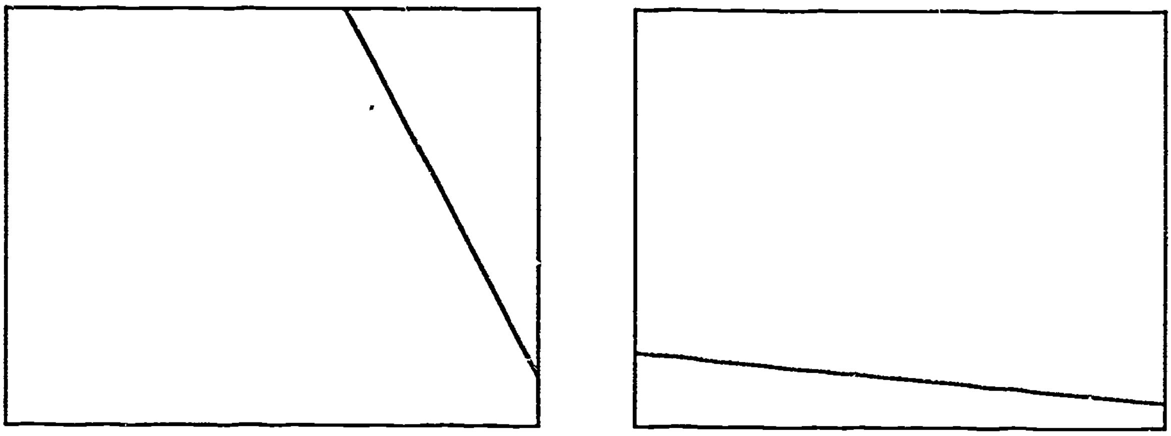
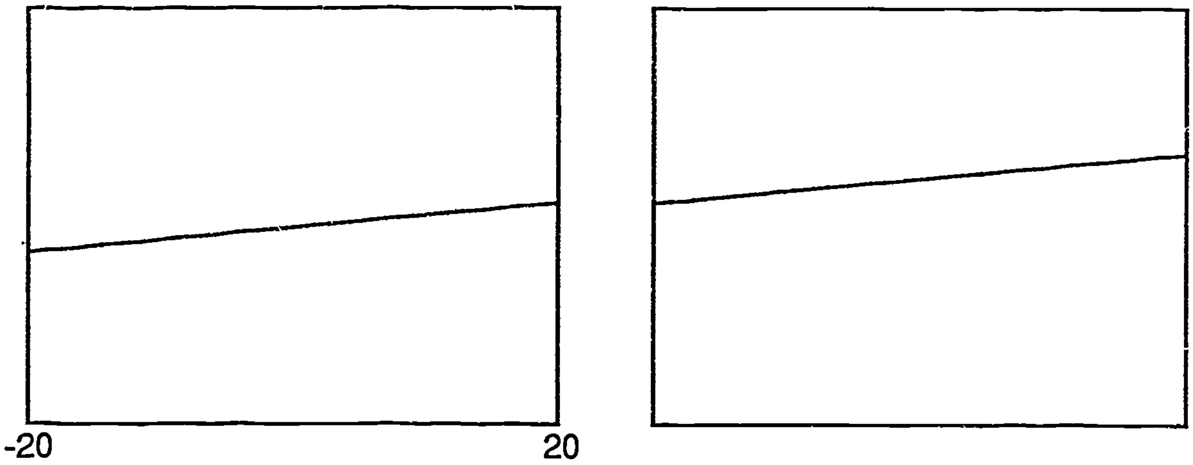


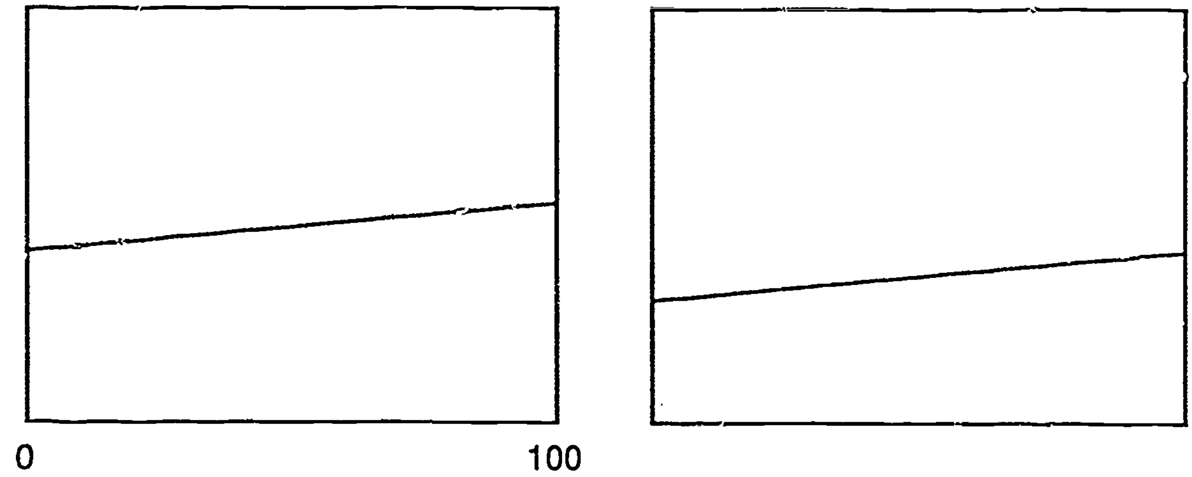
Interviewer Date Student

For this set of problems the horizontal scale on the first window is given. Propose a horizontal scale for the second window so that each could be some the graph of some portion of the same function. If this is impossible in any particular case, write **impossible.**

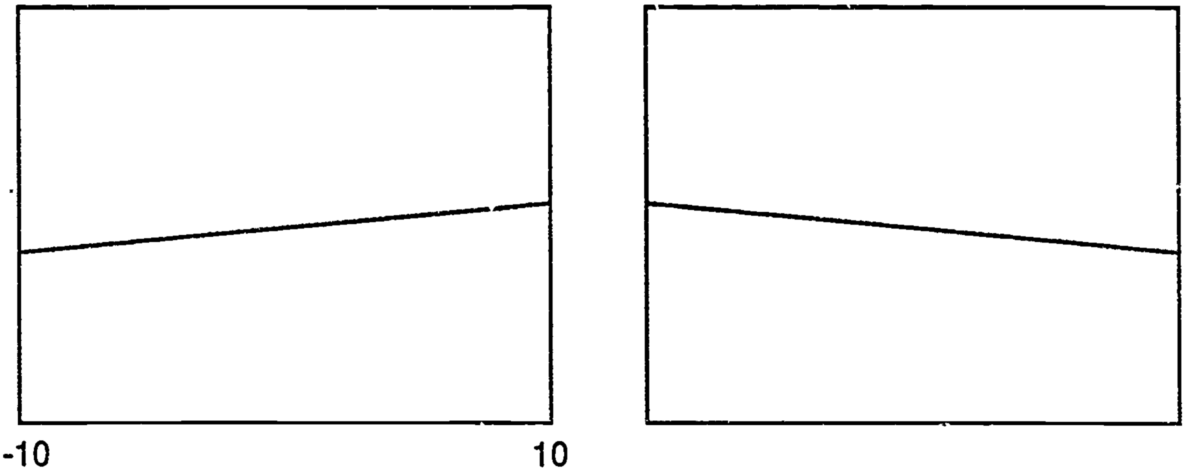
1.



2.



3.



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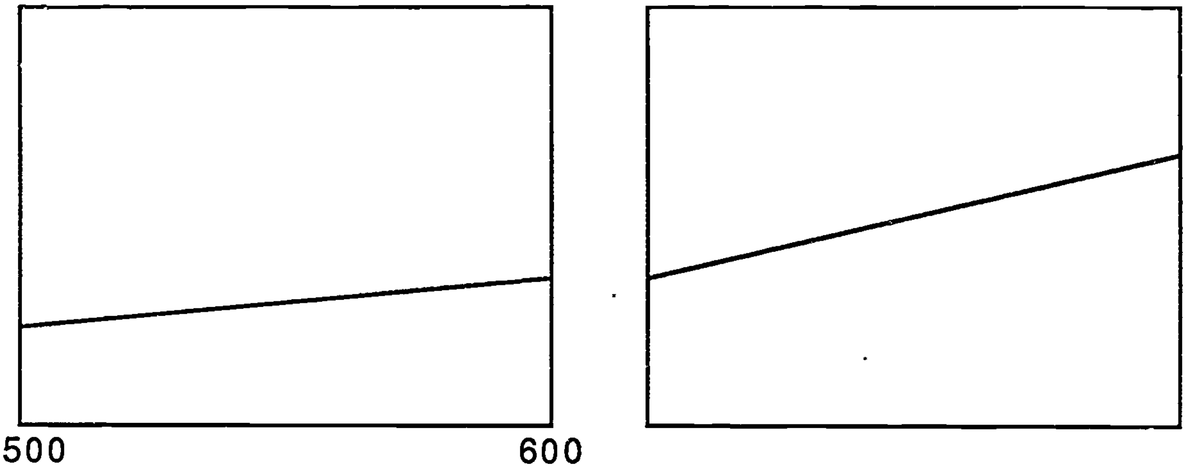

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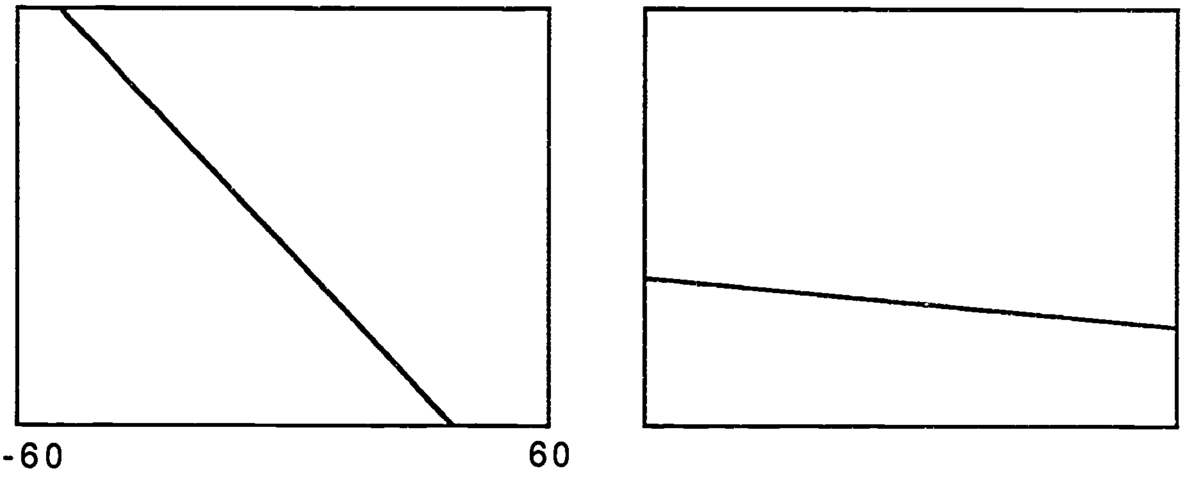
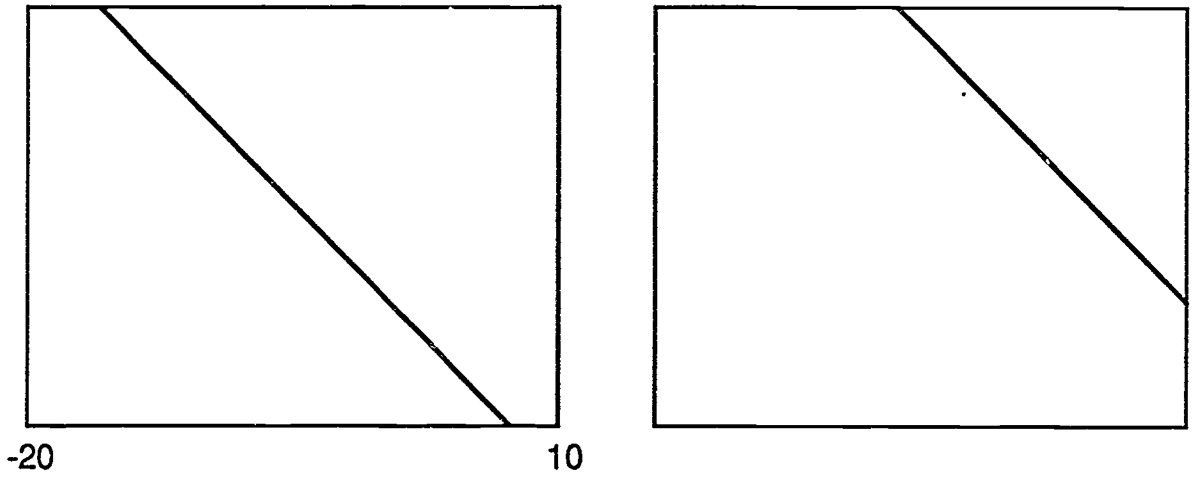


Interviewer Date Student

6.



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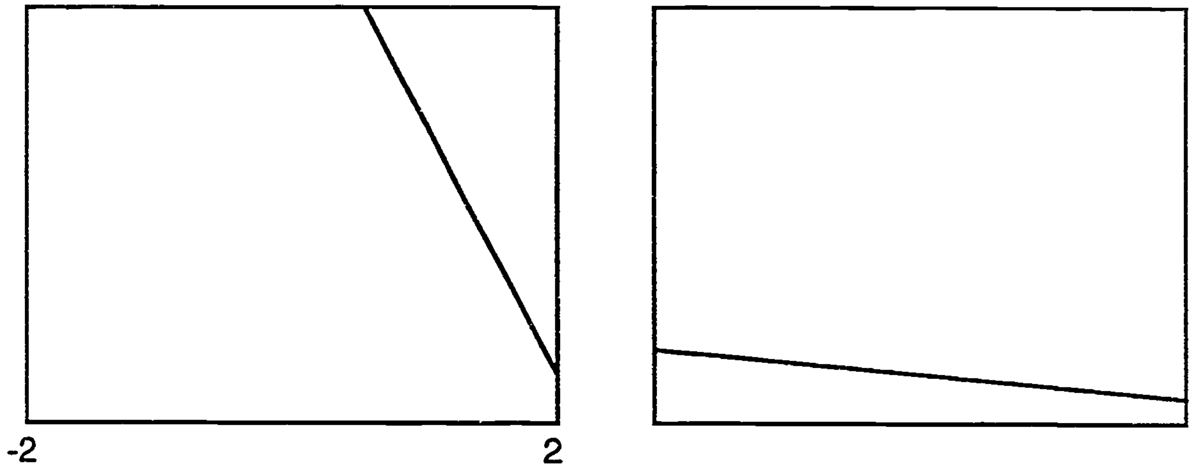
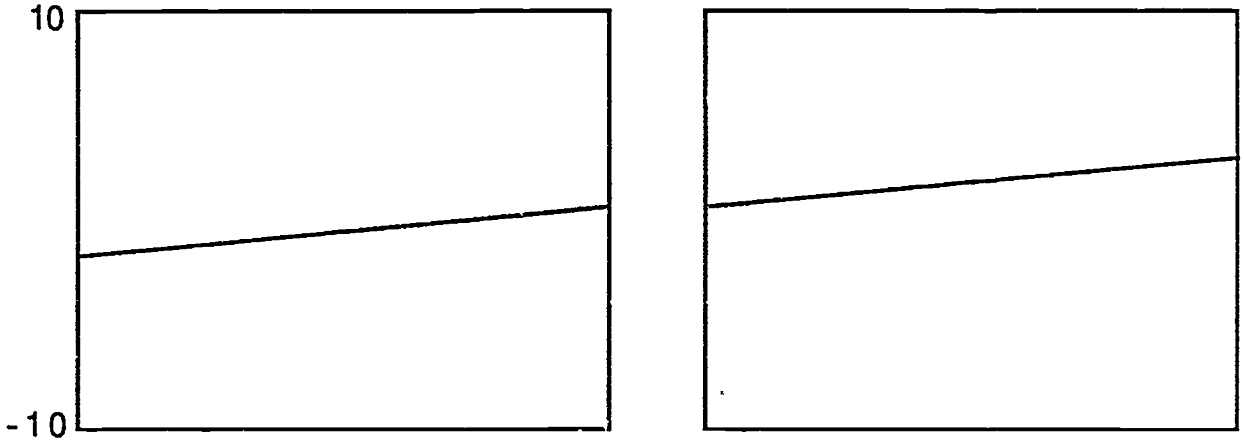


For this set of problems the vertical scale on the first window is given. Propose a new vertical scale for the second window so that each could be the graph of some portion of the same function. If this is impossible in any particular case, write **impossible.**

1.



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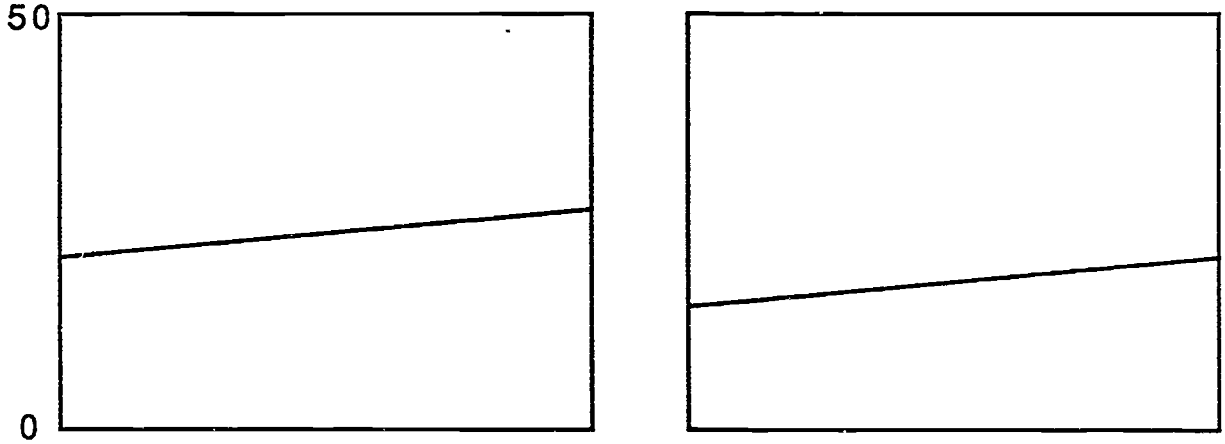
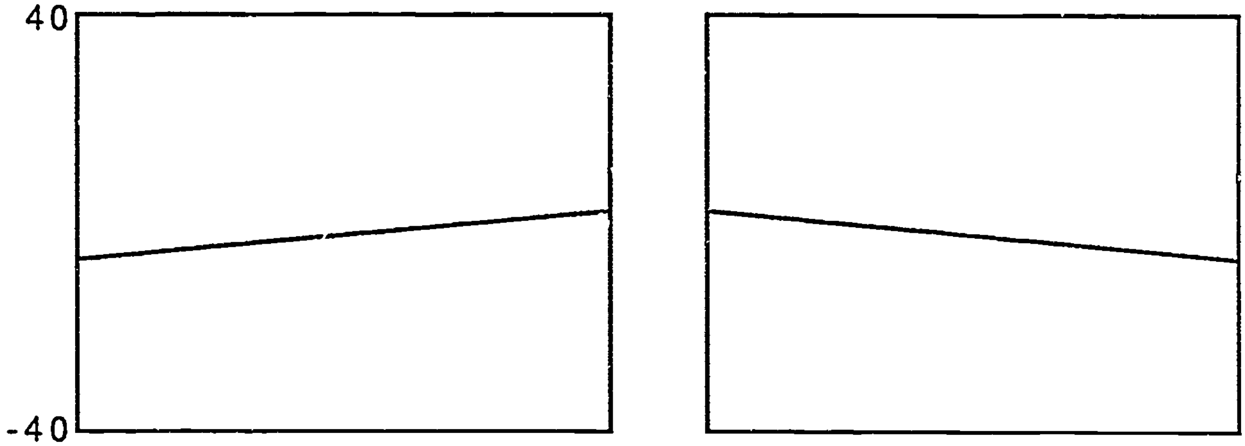


Interviewer Date Student

3.



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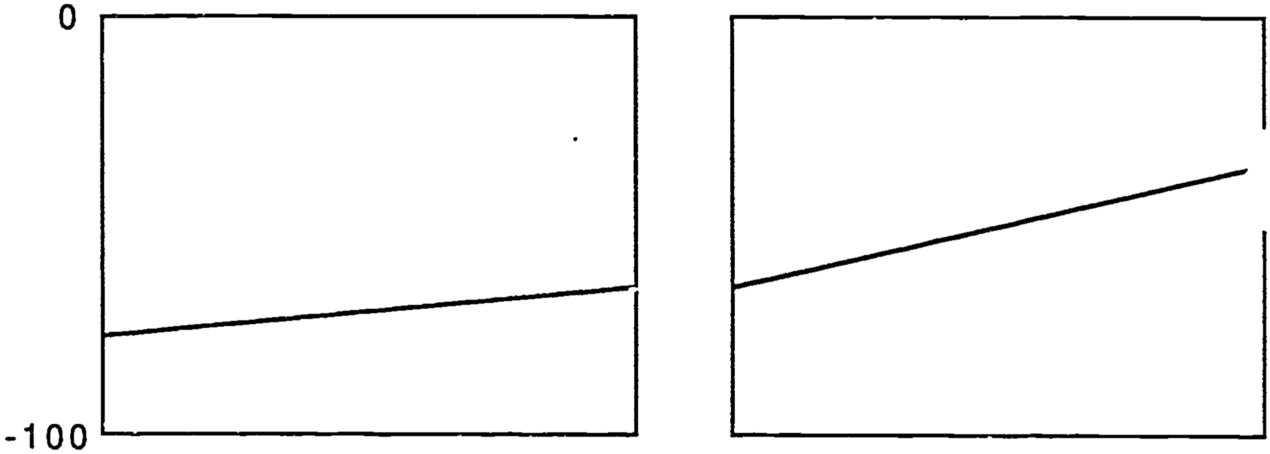

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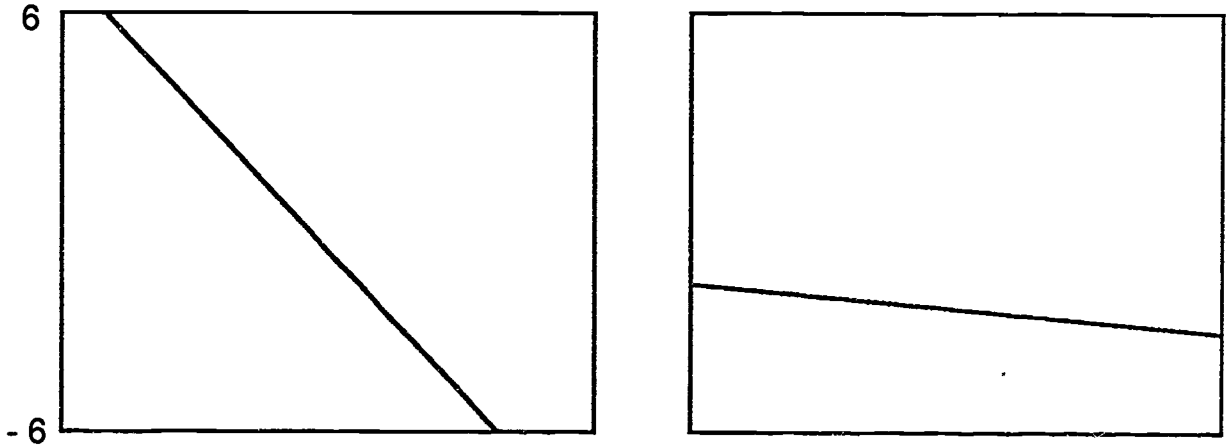

Interviewer Date Student

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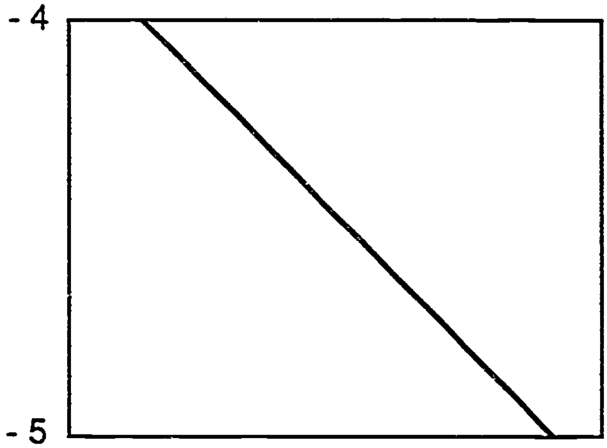

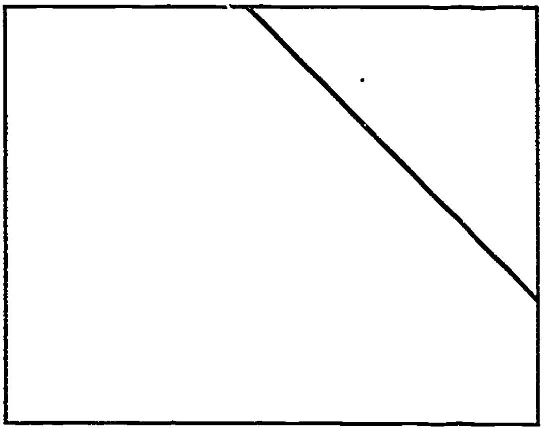

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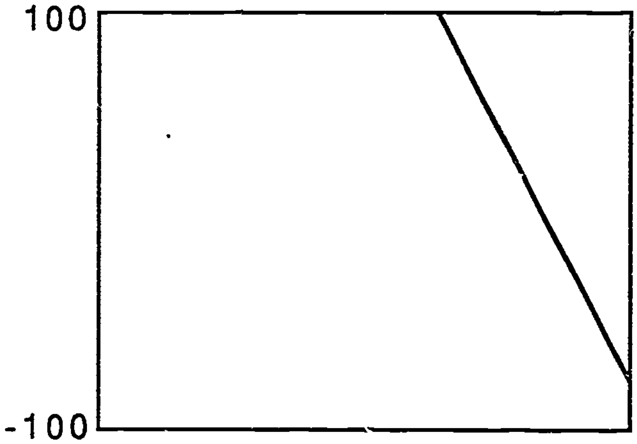
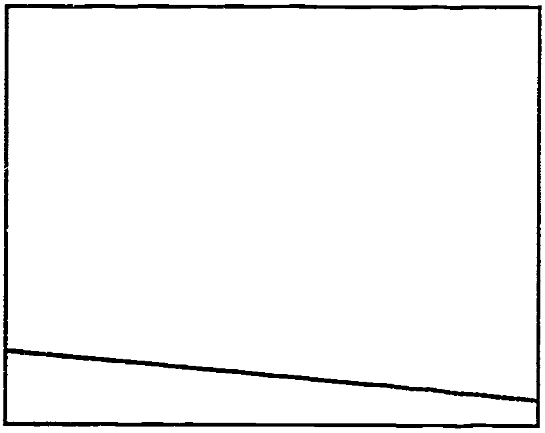




Interviewer Date Student

APPENDIX D

COMPUTER EXPLORING

Object: To determine a natural language for describing scale changes.

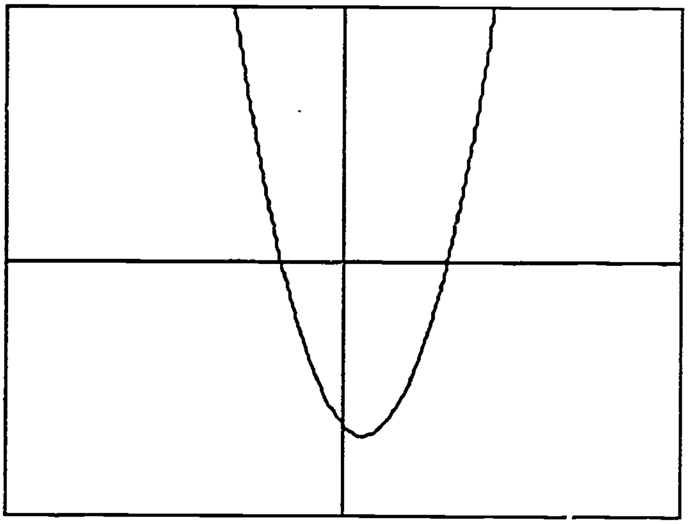
1. *What "handles" do students perceive as being necessary for creating distortions in graphs?*
2. *Does the natural language that they use in describing distortions imply a change of the scale or a change in the function or neither?*

Put up the function $x^2 - x - 4$ *on a 20×20 grid with center origin (-10 to 10 in* x *and* y*).*

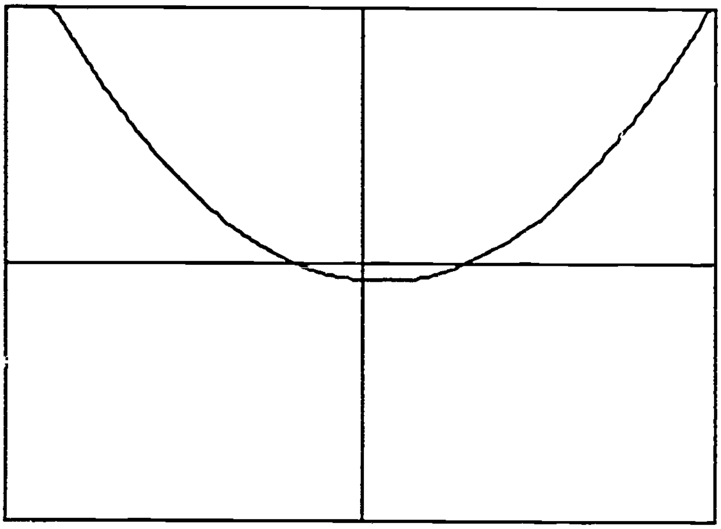
Show the student picture marked "First Change" and ask the student to get the graph to resemble that picture without changing the function.

Repeat the process with remaining pictures.

Initial graph of parabola.



First change.



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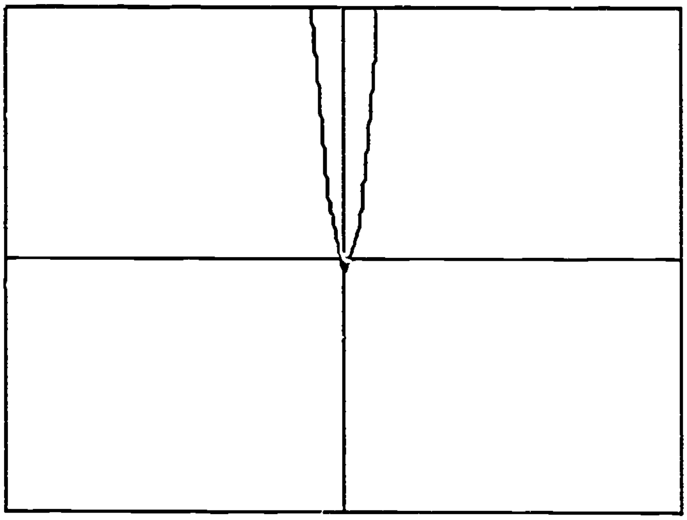
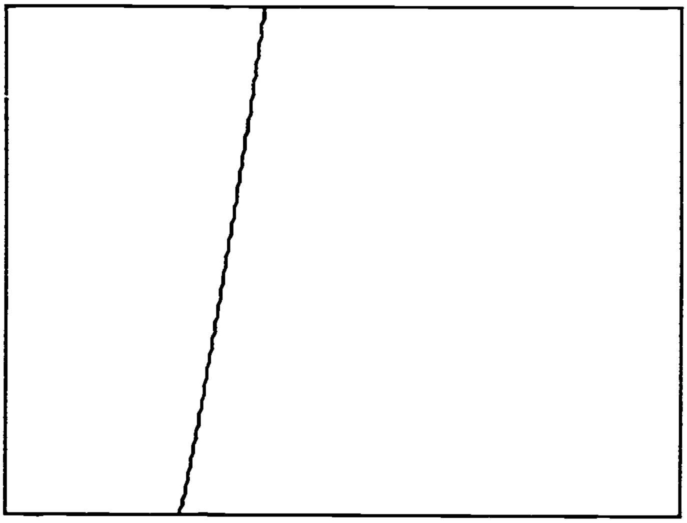
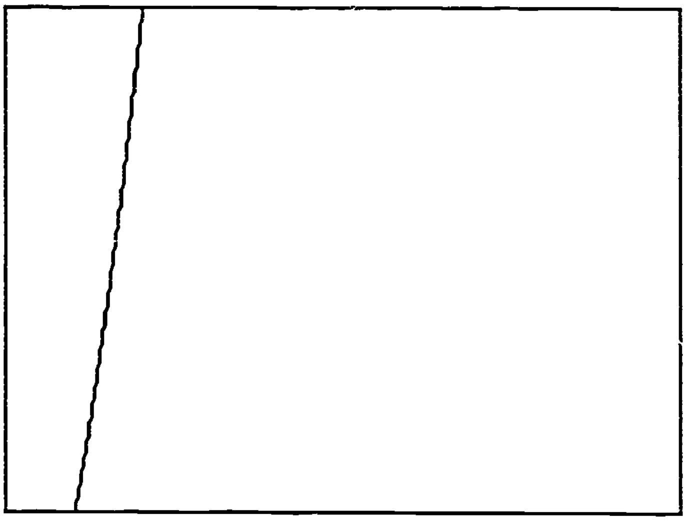
Second change.



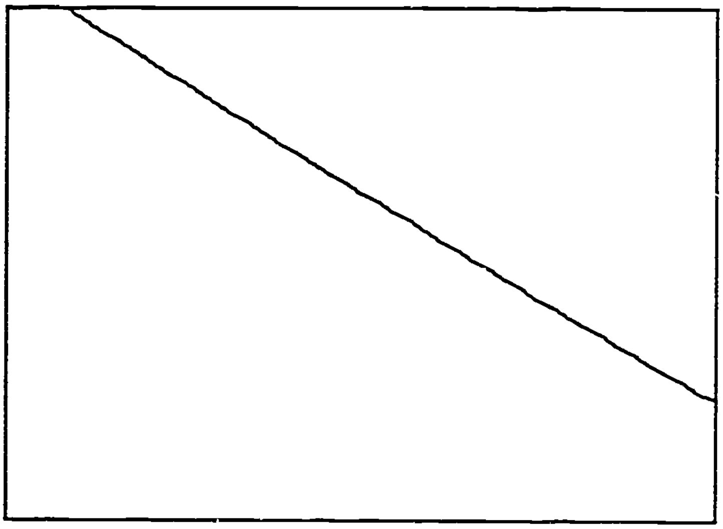
Third change.



Fourth change.



Fifth change.



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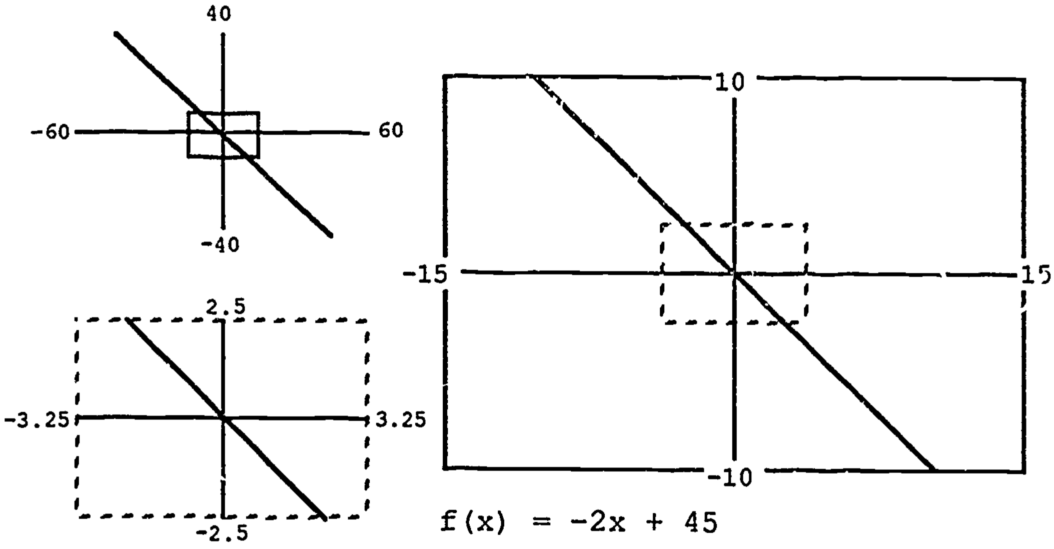
Exploring Scale I

Becoming familiar with the software

1 Press `F` for "Function" and then press `1` to enter your own function.

2 Type `-x` and press `RETURN`.

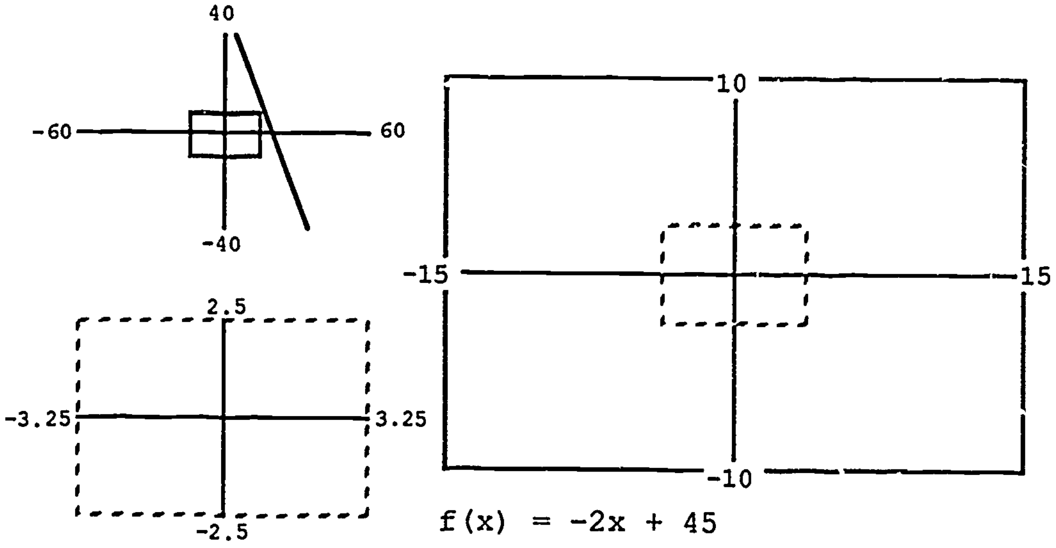

40
-60
60
-40
2.5
-3.25
3.25
-2.5
10
-15
15
-10
f(x) = -2x + 45

Picking a problem

1 Press `F` for "Function" and then press `1` to enter your own function.

2 Type `-2x + 45`

3 Press `RETURN` to accept this function. If you have just started working, your screen should look like this. (If the scale does not match, you should adjust it.)


40
-60
60
-40
2.5
-3.25
3.25
-2.5
10
-15
15
-10
f(x) = -2x + 45

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The little graph in the upper left hand corner shows you that the graph is a line, and shows that the line passes from the first quadrant (the top right quarter of the graph) into the fourth quadrant (bottom right). It looks as though the line, if extended further, would also pass through the second quadrant (the top left).

Why doesn't the line appear on the big graph? The big graph is a magnified view of the central rectangle in the little graph. It shows a region extending from x=-15 to x=15 and from y=-10 to y=10. As you can see on the little graph in the uuper left hand corner, the line does not pass through this central region.

Exploring

4 Think of the big graph as a view through a telescope. Currently, the telescope is not aimed at the line. How would you need to change things to be able to see the line in the telescope? Is the line to the right or to the left of our current view? Is it above or below?

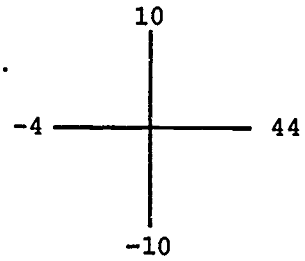
5 You can re-aim the telescope and adjust its magnification to change the region of the graph that you are viewing. To make this change:

a. Press `S` to tell the computer that you want to change the `S`cale of the graph and then `1` to change the coordinates within which you are viewing. This operation can change the way

b. For this first try, let the domain (min to max *x* values viewed) stay as it is and change only the range (*y* values viewed) by changing the upper limit from 10 to 15. RETURN moves you clockwise around the coordinate limits to allow you to choose the space in which you will type a new value.

Press `R` to draw the rescaled shape. Describe the results.

6 Now change the scale again in this way:



How does the image change? Why does it change in this particular way?

7 Change the scale again, this time trying for a view that allows you to see the line as it passes from the second quadrant, through the first, and into the fourth quadrant.

Generalizing

8 Steps 1-3 show you how to pick a new function. Enter this : `-8X+432`.

Like the previous line, this one also passes through all but the third quadrant. Again, change the scale so that you can see "a small but reasonable amount beyond" where the line crosses the X and Y axes.

9 Try several other lines. Find some reliable way to predict an appropriate scale from the coefficients of the function that produces the line.

The method you invent will depend on how big a distance you interpret "a small but reasonable amount beyond" to mean. Any scheme you can defend will be fine.

10 Here are three lines to test your method on. `.01X-500` `500X+0.3` `-3X+400`

All three of these lines do pass through three quadrants, though they may not, at first, appear to. Try to find a scale for the graph that shows quite clearly that the lines start in one quadrant, pass into another, and then pass from that one into still another quadrant.

Theorems and proofs

11 All the lines that I have shown you have passed through exactly three quadrants. Make a reasoned case for your answers to each of the following questions.

Are there lines that pass through only one quadrant? If so, for which quadrant(s)?

Are there lines that pass through exactly two quadrants? If so, for which quadrant(s)?

Are there lines that pass through all four quadrants?

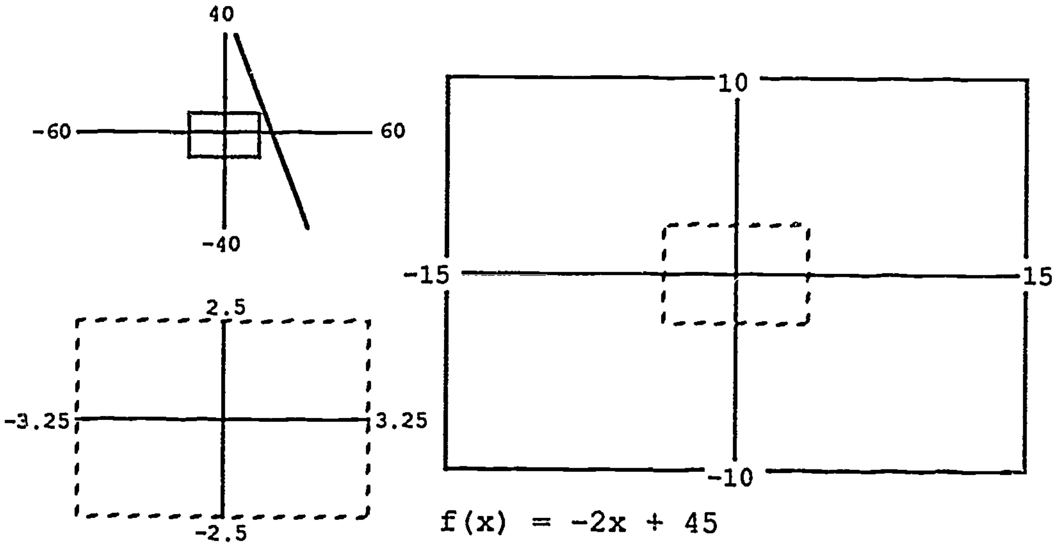
Exploring Scale II

Picking a problem

1 Press `F` for "Function"

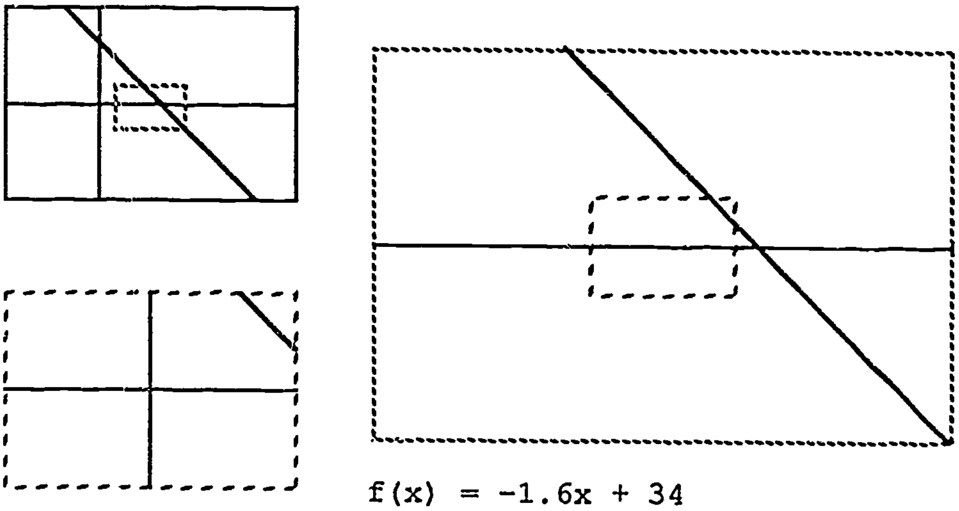
2 Type `-1.6x + 34`

3 Press `RETURN` to accept this function. Your screen should look like this.


40
-60
60
-40
10
-15
15
-10
2.5
-3.25
3.25
-2.5
f(x) = -2x + 45

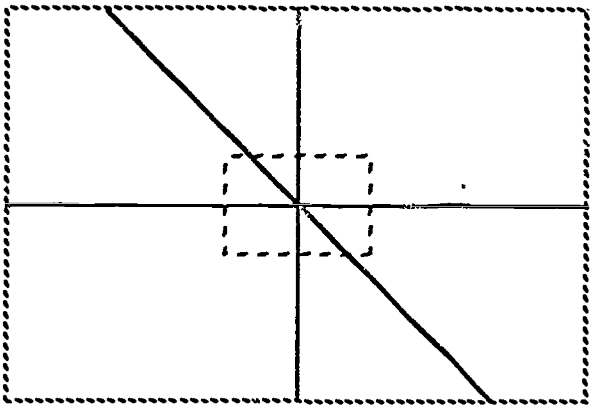
Exploring

4 Change the scale so that the screen looks like this:

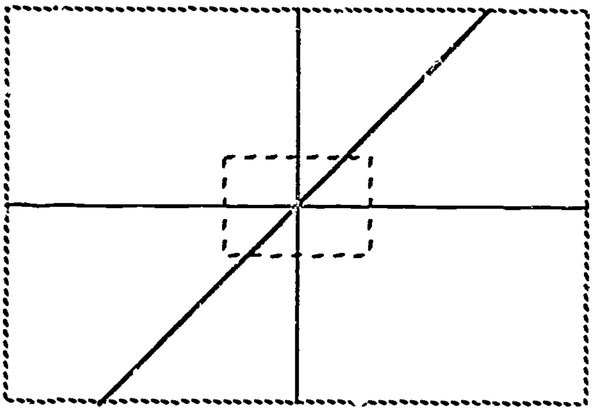


Interviewer Date Student

5 Change the scale again, this time trying to get the screen to look like either of these:



$f(x) = -1.6x + 34$



$f(x) = -1.6x + 34$

Generalizing

6 Pick coefficients for a new line and try again to find a scale that makes your screen look like either of the figures in Exploration 5.

7 Try several other lines. Look for ways to help you predict an appropriate scale from the coefficients of the function. Make sure you try some "odd cases."

Theorems and proofs

8 For some (perhaps all) of your lines, you have found ways to make them *appear* to pass through the center of the graph and through the corners. Yet you know that they do not, in reality, pass through the center. Why would it *appear* so if it *isn't* so?

9 Suppose you were required to keep your graph symmetric. That is, the distance across the graph from left to right (the domain, calculated as Xmax-Xmin) had to be the same as the distance from the top to the bottom of the graph (the range, Ymax-Ymin). Could you still make any line *appear* to pass through the center of the graph? Could you still make any line *appear* to pass through opposite corners of the graph?

Clearly, one cannot try experiments like these with all possible lines. Still, it is possible to make some reasoned arguments about how the experiments would work out. Make some claims about either the symmetric or asymmetric graphs you have been constructing and support them.

Exploring Scale III

Picking a problem

1 Press `F` for "Function."

2 Type `3(x - 6)(x + 2)`

3 Press `RETURN` to accept this function. Make sure you have a scale that shows clearly that this curve passes through all four quadrants.

Picking a new problem

4 Press `F` again and type the polynomial function `3(x - 2)(x - 2)`.

Exploring

5 Change the scale until you are sure you can see clearly what quadrants this curve passes through.

Generalizing

6 Instead of picking a problem by picking roots, create a second order polynomial by picking its coefficients. Try for "strange" cases. For each of your curves, play with scale or values until you can make a clear case about which quadrants the curve passes through.

Theorems and proofs

7 Make a reasoned case for your answers to each of the following questions.

Are there second order polynomials whose graphs pass through only *one* quadrant? If so, for which quadrant(s)?

Are there second order polynomials whose graphs pass through exactly *two* quadrants? If so, for which quadrant(s)?

Are there second order polynomials whose graphs pass through exactly *three* quadrants? If so, for which quadrant(s)?

Exploring Scale IV

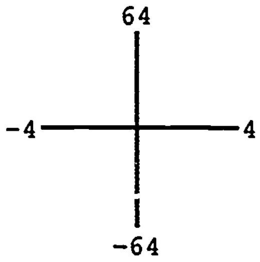
Setting up

1 Press `F` for "Function."

2 Type x^3.

3 Press `RETURN` to accept this function.

4 Use this scale:



5 Tape a transparency on the screen (or, with felt-tip marker, mark directly on the screen) and trace over the graph you see.

Picking a problem and exploring

6 One by one, type in each of these functions:

$x^3 + 13x^2 + 50x + 50$

$x^3 - 7x^2 + 11x + 5$

$-x^3 + 16x^2 - 80x + 128$

$x^3 - 4x^2 + 15x + 18$

$2x^3 - 14x^2 + 22x + 10$

7 Change the scale, if possible, to make your new graph fit the tracing of x^3 exactly.

8 Change the scale, if possible, to make a horizontal line at the height of the constant.

Generalizing

9 Try this with other cubic polynomials.

10 Start with the tracing of x^2 on a scale that runs from -10 to 10 in x and from -100 to 100 in y. Then try to fit arbitrary quadratics to it through scale changes.

11 Recall or retry your experiments with linear functions. Start with a tracing of x on a scale that runs from -20 to 20 in both x and y.

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